Parameter Estimation for Mixed-Mechanism Tear Film Thinning

Rayanne A. Luke[1], Richard J. Braun[1], Tobin A. Driscoll[1], Deborah Antwi[2], Carolyn G. Begley[2]

[1]Department of Mathematical Sciences, University of Delaware, [2]School of Optometry, Indiana University

Introduction

Tear film breakup (TBU) occurs when a dry spot forms on the eye. Evaporation causes relatively slow thinning [1]; lipid-driven tangential flow is hypothesized to cause rapid breakup [2]. We implement a model for a mixture of these mechanisms and fit to breakup occurring on an intermediate time scale and contrast with purely evaporative results. Ranges for parameters affecting tear film (TF) thickness and fluorescent (FL) intensity distributions over time are not well known; our estimates are comparable to experimental values.

Methods

- The tear film is modeled as a single-layer incompressible Newtonian fluid.
- We derive a system of equations for TF thickness, pressure, osmolarity, FL concentration and surfactant concentration via lubrication theory.
- The system is nondimensionalized using an evaporative time scale and discretized in space using Chebyshev spectral collocation, resulting in DAEs (differential algebraic equations) solved via Matlab's ode15s.
- Extracted FL intensity data from every time level across a spot or streak TBU is fit with the model (see Fig. 1) using Matlab's lsqnonlin (Levenberg-Marquardt algorithm).

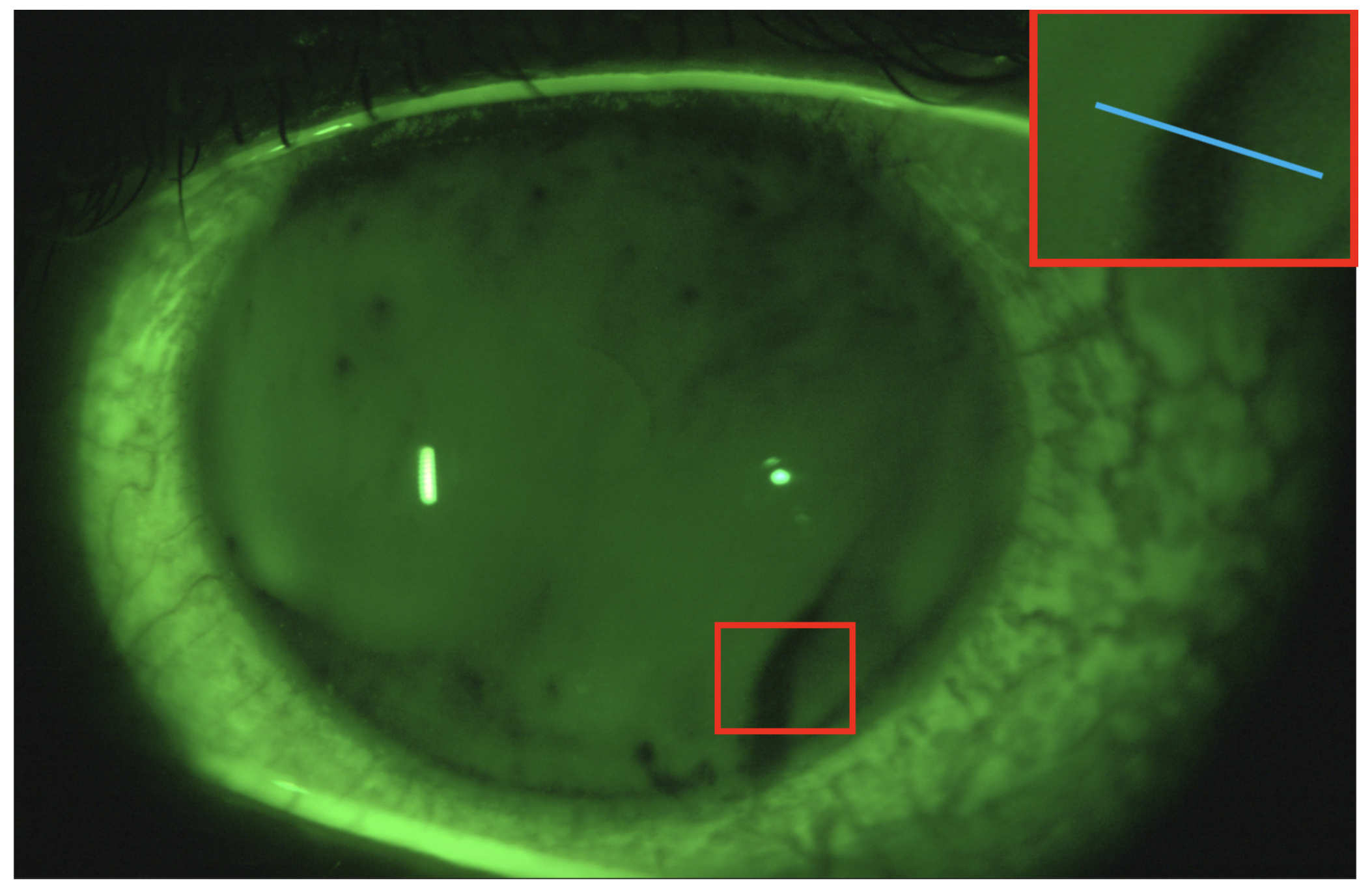


Figure 1: This streak TBU is fit with our Cartesian model.

Goals

- Determine experimental quantities that cannot be measured *in vivo* during TBU by parameter identification via fitting to our models.
- Compare our mixed-mechanism and evaporative thinning models [3].

Cartesian Model

We model TF thickness, $h(x,t)$, pressure, $p(x,t)$, osmolarity, $c(x,t)$, FL concentration, $f(x,t)$, and surface concentration of lipid $\Gamma(x,t)$, and calculate FL intensity $I(h,f)$:

$$\partial_t h = -\partial_x (h\bar{u}) + P_c(c-1) - J, \quad 0 < x < X_0$$

$$\partial_t \Gamma = \left[\mathrm{Pe}_s^{-1}\partial_{xx}\Gamma - \partial_x(u_s\Gamma)\right] B$$

$$h(\partial_t c + \bar{u}\partial_x c) = \mathrm{Pe}_c^{-1} h\partial_{xx}c - P_c(c-1)c + Jc$$

$$h(\partial_t f + \bar{u}\partial_x f) = \mathrm{Pe}_f^{-1} h\partial_{xx}f - P_c(c-1)f + Jf$$

$$p = -\partial_{xx}h, \quad I = I_0\frac{1-e^{-\phi h f}}{1+f^2}$$

where J is evaporation, $\bar{u}$ is the depth-averaged fluid velocity, u_s is the surface velocity, B is a smooth transition function, and ϕ, P_c, Pe_s, Pe_c, and Pe_f are constants. No flux at $x=0$, $x=X_0$.

Optimization

$$\arg\min_{v, X_I, (\Delta\sigma)_0} ||I_{th}(x,t) - I_{ex}(x,t)||^2,$$

where the parameters are

- v, rate of evaporation (in μm/min),
- X_I, width of glob (in mm),
- $(\Delta\sigma)_0$, change in surface tension (in μN/m),

with experimental FL intensity I_{ex} (Fig. 1) and I_{th} computed from our Cartesian model.

We use three choices for the evaporation J: zero, uniform, and high under the glob, zero outside.

Acknowledgements: Funded by NSF-DMS Grant 1909846 (Braun) and NIH grant 1R01EY021794 (Begley).

Nondimensional Model Solutions

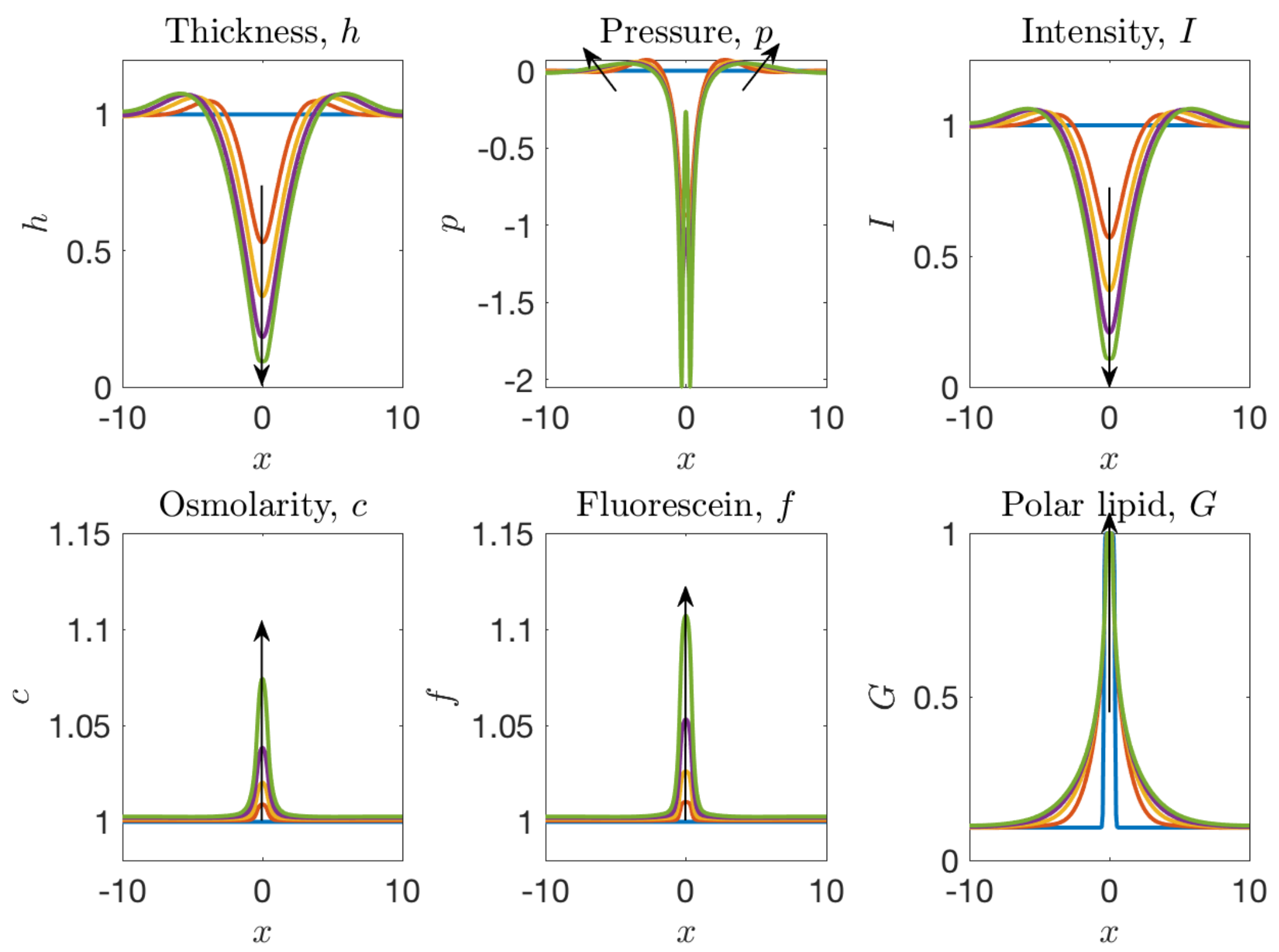


Figure 2: A surface tension based time scale is used. Tangential flow drives extreme thinning in the streak center. Similarities between I and h allow inferences about h from exp. intensity data. Arrows show increasing time (final time: 0.2 s).

Mixed-Mechanism Fit Results

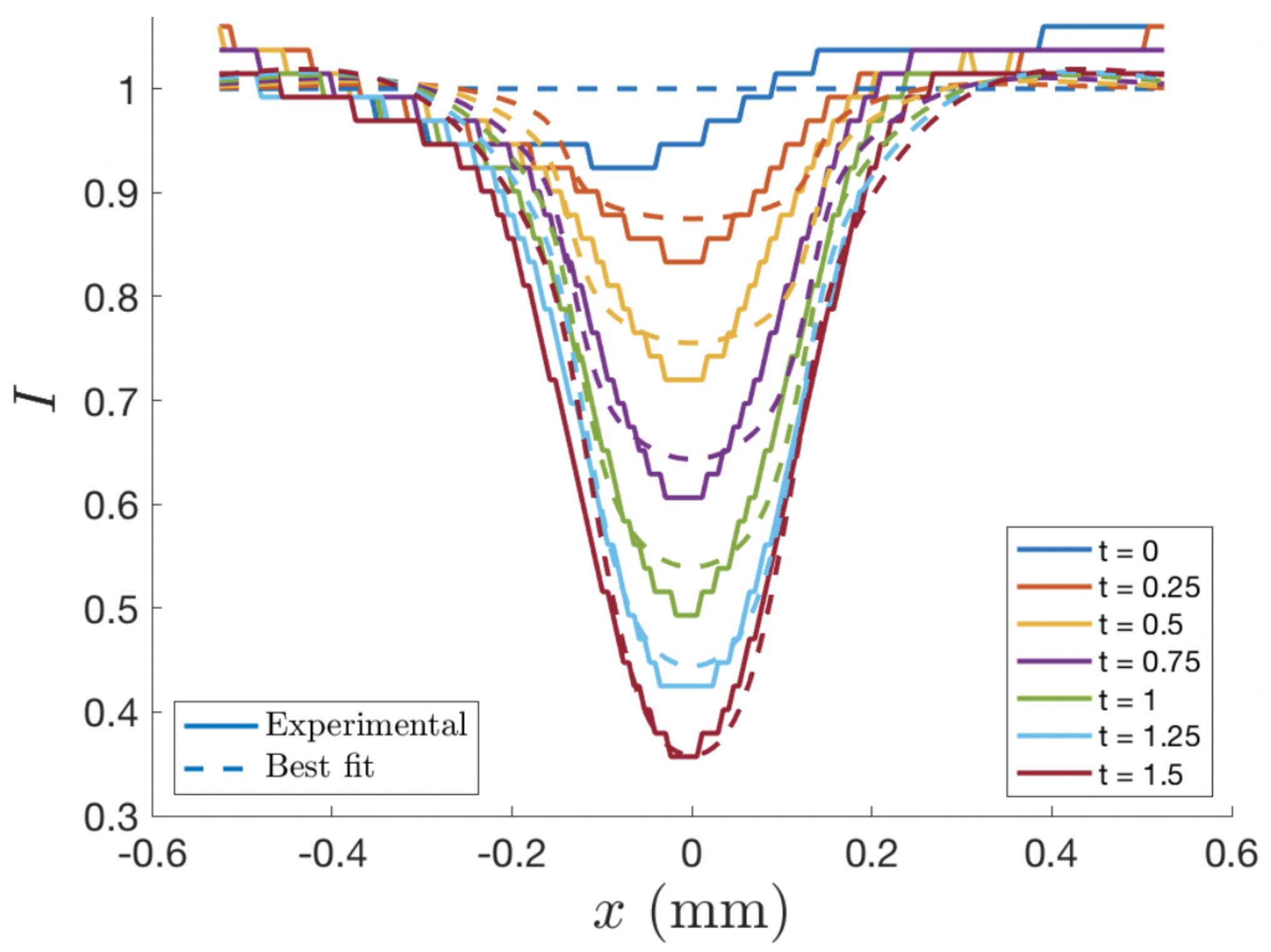
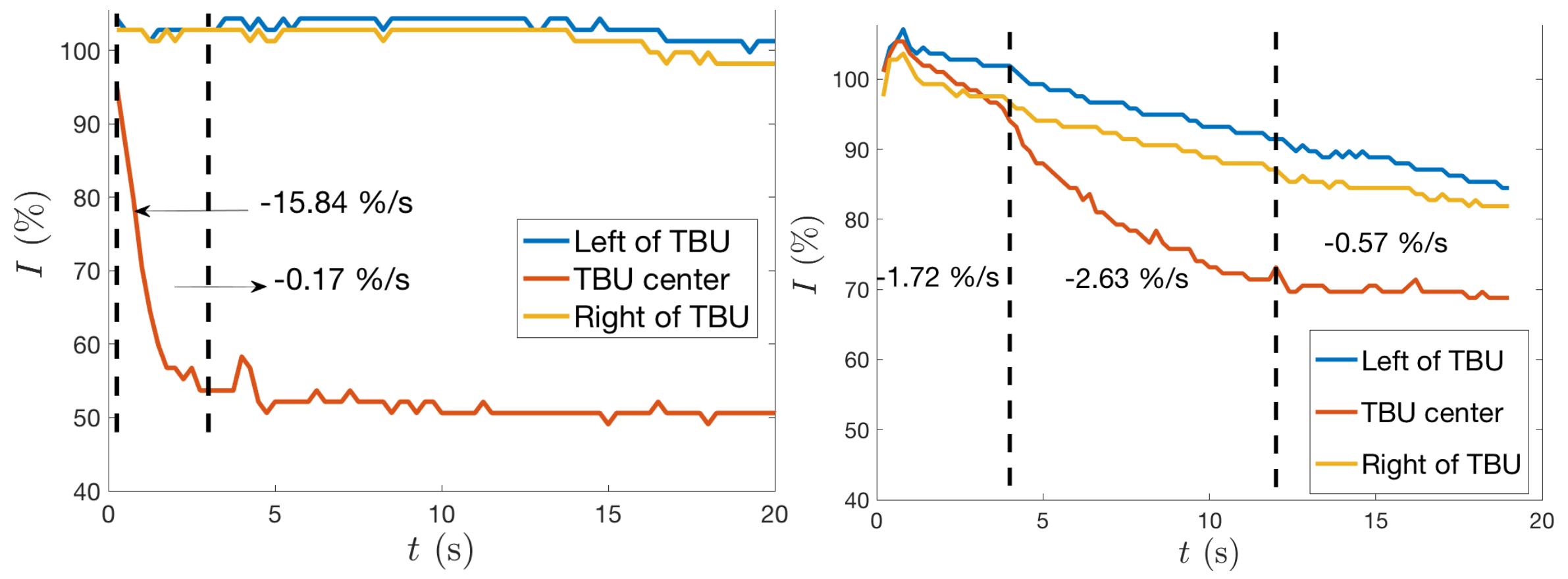


Figure 3: Intermediate streak TBU (Figs. 1, 4a). An evaporative time scale and the 3rd evaporation choice are used.

Mixed-Mechanism vs. Evaporation

Evaporation acts on a longer time scale than tangential flow driven by surface tension gradients, shown by the less rapid decrease in FL intensity in Fig. 4b.



(a) (b)

Figure 4: (a): intermediate streak shown in Figs. 1 and 3, (b): evaporative streak from same subject and visit (Fig. 5).

Evaporative Fit Results

Evaporation is a Gaussian with width x_w controlled by peak rate $v_{\max}$ and constant background rate $v_{\min}$.

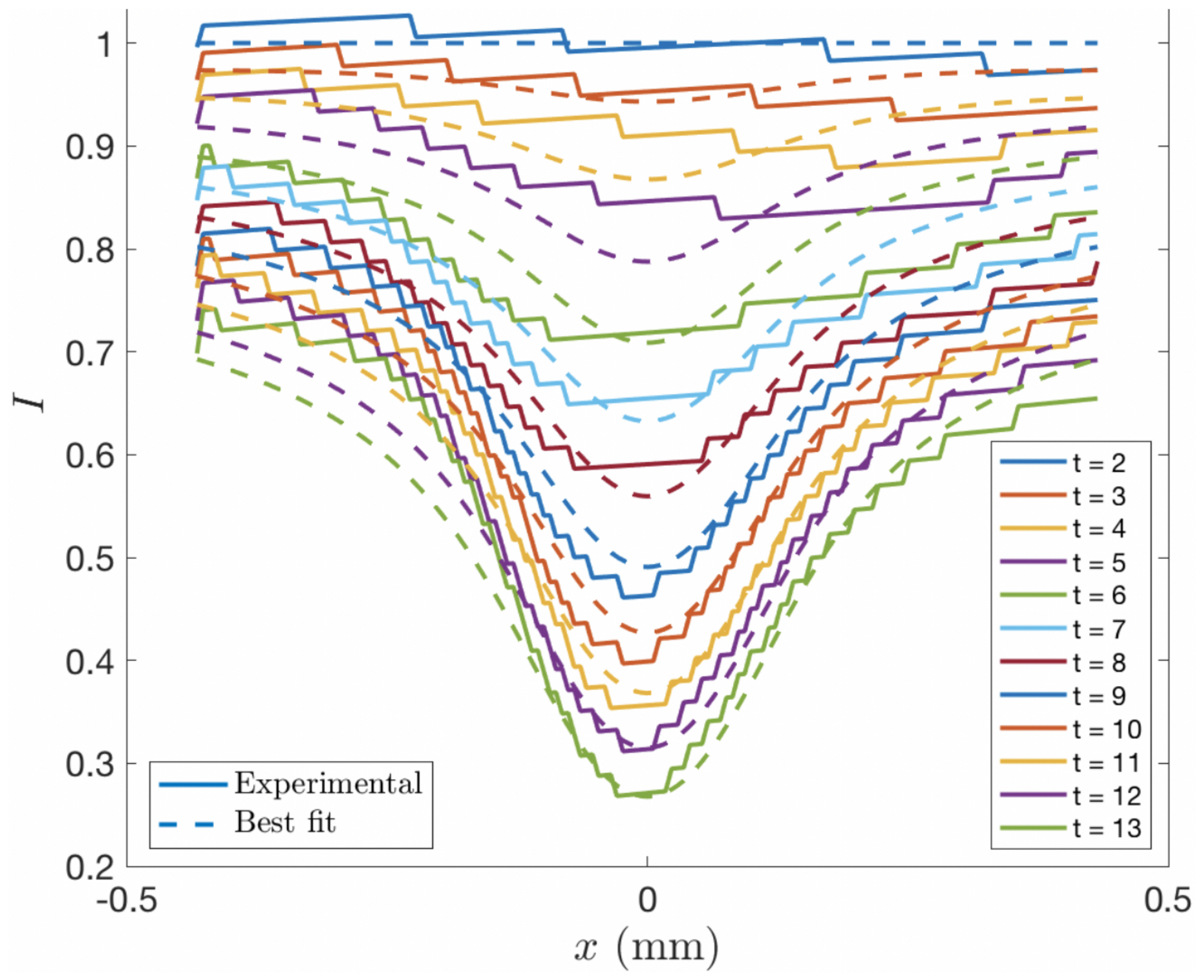


Figure 5: Evaporation-driven streak TBU (Fig. 4b).

Comparison of Fig. 3 and Fig. 5

TBU Type	Max evap rate $(\frac{\mu m}{min})$	Min evap rate $(\frac{\mu m}{min})$	glob/ Gaus width (mm)	Δ surf tens $(\frac{\mu N}{m})$	Init TF ht (μm)	Init FL conc (%)
mix	35.1	0	0.140	1.96	2.91	0.4
evap	20.3	5.32	0.0901	N/A	3.86	0.24

Conclusions & Next Steps

- Intermediate fits are improved with nonzero evaporation, evidence that tangential flow and evaporation cooperate to cause breakup.
- Most instances of TBU are mixed-mechanism and the model helps decipher how much of each is present.
- We are fitting more cases of rapid lipid-driven thinning and will move to 2D models.

References

[1] King-Smith PE, Hinel EA, Nichols JJ (2010) Application of a novel interferometric method to investigate the relation between lipid layer thickness and tear film thinning. *Invest Ophthalmol Vis Sci.* **51**(5):2418-2423

[2] Zhong L, Braun RJ, Begley CG, King-Smith PE (2019) Dynamics of fluorescent imaging for rapid tear thinning. *Bull Math Biol.* **81**(1):39-80

[3] Braun, R. J., et al. (2018) On tear film breakup (TBU): dynamics and imaging. *Math. Med. Biol.* **35**, 145-180.