Graph Signal Processing: Classifying Window Functions of Cayley Graphs using Representation Theory

Kris Hollingsworth

Advisors: Mahya Ghandehari and Dominique Guillot

Department of Mathematical Sciences, University of Delaware, Newark, DE, USA

Introduction

A graph signal is a function $f : V \to \mathbb{C}$, where V is the vertex set of a graph. With increasing amounts of data being recorded which naturally embed in a graph structure, there is growing interest in generalizing tools of classical signal processing to this setting. See [1] for a discussion of generalizing the DFT to this setting. For an introduction more focused on generalizing traditional signal processing applications, see [2].

Today I discuss recent work on the problem of characterizing graph signals with well-localized translations which arose in generalizing the short-term Fourier transform. In particular, I explain the use of representation theory to describe these functions for many families of Cayley graphs.

Graph Signal Fourier Transform

The graph Laplacian is $\mathcal{L} = D - A$, with D the diagonal degree matrix and A the standard adjacency matrix. If we fix a basis of eigenvectors Φ of $\mathcal{L}$, then the Fourier transform of a graph signal f is the expansion of f in terms of Φ. That is, for a graph of order N,

$$\widehat{f}(\lambda_i) = \langle f, \phi_i \rangle_{\ell^2(G)} = \sum_{n=1}^{N} f(v_n)\overline{\phi_i(v_n)}.$$

In this setting, the inverse Fourier transform is given by

$$f(v_n) = \sum_{i=0}^{N-1} \widehat{f}(\lambda_i)\phi_i(v_n).$$

If we think of f and $\widehat{f}$ as column vectors and Φ as the matrix of basis vectors, then these definitions naturally lend themselves to the notation

$$\widehat{f} = \Phi^* f, \qquad \text{and} \qquad f = \Phi \widehat{f}.$$

Graph Signal Translation

The graph translation operator is defined by convolution with the Kronecker delta function δ_ℓ and then by taking the inverse Fourier transform:

$$(T_\ell f)(v_k) = \sqrt{N}(f * \delta_\ell)(v_k) = \sqrt{N}\sum_{i=0}^{N-1} \widehat{f}(\lambda_i)\overline{\phi_i(v_\ell)}\phi_i(v_k).$$

Remark: One of the chief difficulties in the graph setting is the lack of regularity in both the vertex and spectral domains. For example, in general, the graph translation operator is not an isometry.

Cayley Graphs

Let $\mathcal{G}$ be a group. Let $S \subseteq \mathcal{G}$. Then the Cayley graph $\mathsf{Cay}(\mathcal{G}, S)$ is a graph (V, E) whose vertices are indexed by the elements of $\mathcal{G}$, with adjecency defined as

$$E(x, y) = \begin{cases} 1, & \text{if } x^{-1}y \in S \\ 0, & \text{if } x^{-1}y \notin S. \end{cases}$$

Window Functions and Previous Work

Let G be a graph of order N. A window function is $f : V \to \mathbb{C}$ such that $T_i f(v_k) = 0$ when $d(v_i, v_k) > r$ for some integer $0 < r < N$. Here we use the geodesic distance as our metric.

Previous Work: The authors prove in [3] that if $\widehat{f}$ is a polynomial of degree r, then f will be a window function. This shows that the dimension of the space of window functions is at least $r + 1$.

Our Contribution: Using representation theory, we fully classify window functions on Cayley graphs generated by the union of conjugacy classes. To do this, we exploit the fact that these graphs have an eigenbasis formed by the coordinate functionals of the group's irreducible representations. These details are provided in the next column.

New Theorem

Let $\mathcal{G} = \{g_i\}_{i=1}^{|\mathcal{G}|}$ be a finite group. Let $G = \mathsf{Cay}\,(\mathcal{G}, S)$ be the Cayley graph generated by $S \subseteq \mathcal{G}$ where $S = \bigcup_{i \in I} C_i$ and each C_i is a conjugacy class. Let e denote the identity element, and let $\{h_i\}_{i=1}^{r}$ be a complete set of representatives for the conjugacy classes of $\mathcal{G}$. Let χ be the standard character table of $\mathcal{G}$ with columns ϕ_i corresponding to the conjugacy class containing h_i. Also denote the characters by $\{\chi_j\}_{j=1}^{r}$. Then f is a window function with $T_i f(v_t)$ sharply localized in the ball of distance k centered at vertex v_t if and only if $d(e, g_i) > k$ implies that $\langle f, \phi_i \rangle = 0$.

Further, given the subset $S_k = \{i \in I \mid d(e, h_i) > k\}$, we can construct an orthogonal basis for the space of window functions on G localized in the k-ball by lifting the vectors $\{\phi_i\}_{i \in S_k}$ from $\mathbb{C}^{|\widehat{\mathcal{G}}|}$ to $\mathbb{C}^{|\mathcal{G}|}$ and then taking their Fourier inverse. That is, $T_m f(v_t) = 0$ for all $d(v_m, v_t) > k$ if and only if $\widehat{f}$ satisfies the span condition given to the right. Note that $\chi_t(e)$ is the degree of the representation χ_t, and each block is size $\chi_t(e)^2$ by 1.

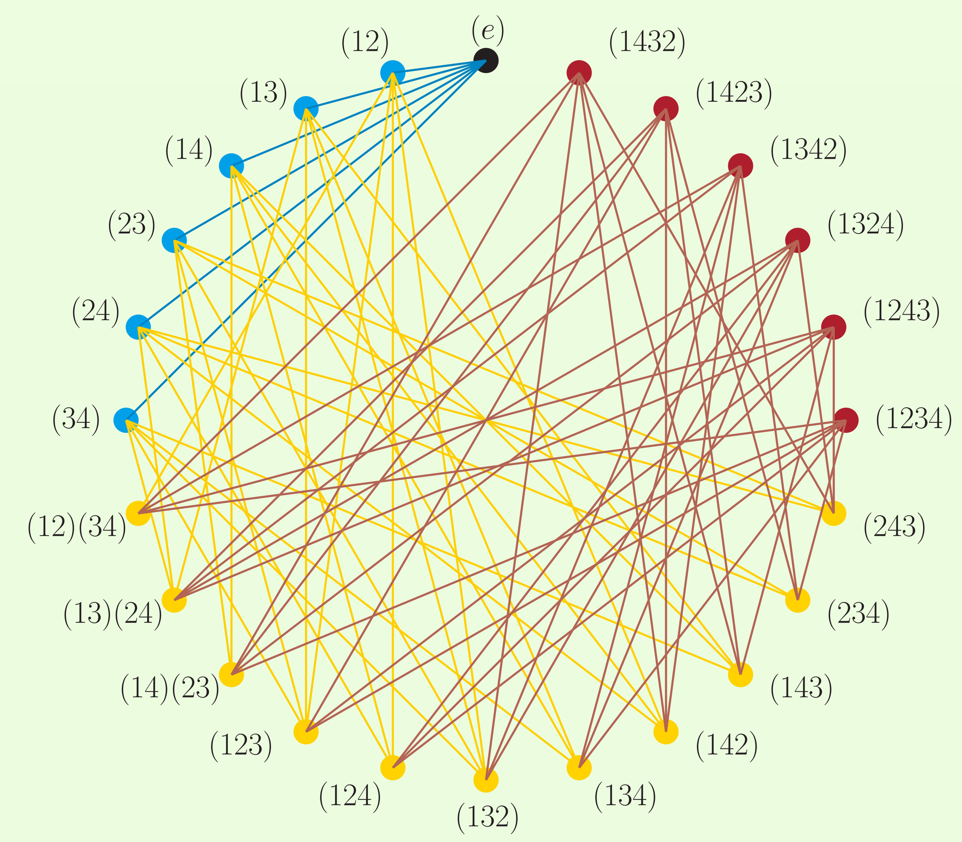
$$\widehat{f} \in \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} \frac{\chi_1(h_i)}{\chi_1(e)} \\ \hline \frac{\chi_2(h_i)}{\chi_2(e)} \\ \hline \vdots \\ \hline \frac{\chi_r(h_i)}{\chi_r(e)} \end{pmatrix} \right\}_{i \in S_k}$$

Main Idea of Proof: Given the basis of coordinate functionals, we can rewrite the translation operator as

$$(T_\ell f)(v_k) = \frac{1}{\sqrt{|\mathcal{G}|}} \sum_{\pi \in \widehat{G}} \pi(e)\widehat{f}(\lambda_\pi)\chi_\pi(\ell^{-1}k),$$

which reduces the problem to finding orthogonality relationships in the character table of the underlying group.

An Example: The Cayley Graph of S_4 Generated by Transpositions



Distance from e	0	1	2	3
Conjugacy Class	e	(12)	(12)(34), (123)	(1234)

The graph $G = \mathsf{Cay}(S_4, \{(12), (13), (14), (23), (24), (34)\})$ is pictured to the left. The character table of S_4 is given in the next column. Let's say that we want to find a basis for all functions $f : V \to \mathbb{C}$ on this graph such that $T_i f(v_k) = 0$ if $d(v_i, v_k) > 2$. Then $\widehat{f}$ must be orthogonal to the last column of the character table in $\mathbb{C}^5$, but we can lift this to $\mathbb{C}^{24}$ using the formula above to determine that

$$\widehat{f} \in \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1^{(1)} \\ 1^{(1)} \\ 1^{(4)} \\ 1^{(9)} \\ 1^{(9)} \end{pmatrix}, \begin{pmatrix} 3^{(1)} \\ -3^{(1)} \\ 0^{(4)} \\ 1^{(9)} \\ -1^{(9)} \end{pmatrix}, \begin{pmatrix} 3^{(1)} \\ 3^{(1)} \\ 3^{(4)} \\ -1^{(9)} \\ -1^{(9)} \end{pmatrix}, \begin{pmatrix} 2^{(1)} \\ 2^{(1)} \\ -1^{(4)} \\ 0^{(9)} \\ 0^{(9)} \end{pmatrix} \right\}.$$

Here we use the superscript notation $^{(n)}$ to denote that the size of the blocks are n by 1.

Eigenvectors of the Graph Laplacian

Let $G = \mathsf{Cay}(\mathcal{G}, S)$ with adjacency matrix A. Then for $\rho \in \widehat{G}$

$$A\rho_{i,j} = \lambda_\rho \rho_{i,j},$$

where $\rho_{i,j}$ is the coordinate functional for the representation ρ and $\lambda_\rho = \frac{1}{\rho(e)}\sum_{g \in S}\chi_\rho(g)$. We are able to apply this result to the graph Laplacian as it is a polynomial of the adjacency matrix.

Character Table of S_4

	e	(12)	(12)(34)	(123)	(1234)
χ_ι	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1
χ_π	2	0	2	-1	0
χ_{std}	3	1	-1	0	-1
$\chi_{\mathsf{sgn\cdot std}}$	3	-1	-1	0	1

Remark: Using the character table instead of the coordinate functionals when computing the space of window functions yields a substantial decrease in complexity.

Comparison of Results

The theorem in [3] shows that for S_4 generated by conjugacy classes, f is sharply localized if $\widehat{f}$ is a polynomial, or equivalently,

$$\widehat{f} \in \mathrm{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1^{(1)} \\ 1^{(1)} \\ 1^{(4)} \\ 1^{(9)} \\ 1^{(9)} \end{pmatrix}, \begin{pmatrix} 6^{(1)} \\ -6^{(1)} \\ 0^{(4)} \\ 2^{(9)} \\ -2^{(9)} \end{pmatrix}, \begin{pmatrix} 36^{(1)} \\ 36^{(1)} \\ 0^{(4)} \\ 4^{(9)} \\ 4^{(9)} \end{pmatrix} \right\}.$$

It is easy to verify that this forms a 3-dimensional subspace of the 4-dimensional space of window functions for this graph found using representation theory. It is also worth noting that the proof technique used in [3] provides no means of generalization to finding other sharply localized functions.

Future Work

Continuing work on this project will include

- classifying window functions for Cayley graphs with arbitrary generating sets.
- studying the relationship between the Discrete Fourier Transform and the Graph Fourier Transform. (In the case of finite abelian groups, they are identical!)
- generalizing the results from finite groups to locally compact groups.

References

[1] A. Sandryhaila and J. M. F. Moura "Discrete signal processing on graphs," *IEEE Transactions on Signal Processing*, vol. 61, no. 7, pp. 1644–1656, 2013.

[2] D. I. Shuman, S. K. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains," *IEEE Signal Processing Magazine*, vol. 30, no. 3, pp. 83-98, May 2013.

[3] D. I. Shuman, B. Ricaud, and P. Candergheynst, "Vertex-frequency analysis on graphs," *Applied and Computational Harmonic Analysis*, vol. 40, no. 2, pp. 260–291, 2016.