Determination of material parameters of cancellous bone from acoustic measurements in a water tank

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Introduction

Osteoporosis is characterized by a decrease in strength of the bone matrix. This is a serious disease affecting an increasing number of the aged and it is also a threat for potential astronauts.

Since the loss of bone density and the destruction of the bone microstructure is most evident in osteoporotic cancellous bone, which consists of trabeculae and marrow, it is natural to consider the possibility of developing accurate ultrasound models for the sonification of cancellous bone. It would be of enormous clinical advantage if accurate methods could be developed using ultrasonic interrogation to diagnose osteoporosis and bone fractures.

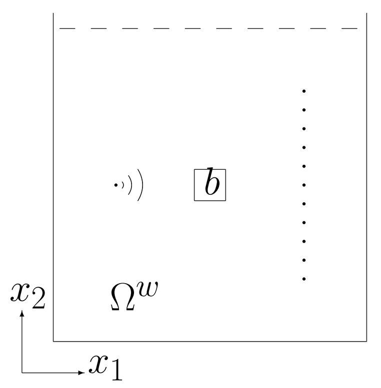

Figure 1: Sketch of the experimental setup.

Green's Function for a Finite Tank

A bone specimen is placed in an open rectangular water tank (Figure 1). In the water region Ω^w, the fluid pressure P and fluid displacement $\mathbf{U}^w$ satisfy

$$-\nabla^2 P - \kappa_0^2 P = f\,, \qquad (1)$$

$$\nabla P - \rho^w \omega^2 \mathbf{U}^w = \mathbf{0}\,, \quad \mathbf{x} \neq \mathbf{x}_0\,, \qquad (2)$$

where $f(\mathbf{x},\mathbf{x}_0) = -q\,\delta(\mathbf{x},\mathbf{x}_0;\kappa_0)$ represents a point source of strength q located at $\mathbf{x} = \mathbf{x}_0$, and κ_0 is the wavenumber of the signal.

To reduce the Helmoltz's equation to a boundary integral equation defined along the interface bwteen bone and water, we construct an infinite series representation of the Green's function for the open tank by the method of images

$$\begin{aligned} G = i\pi \sum_{\ell,m=-\infty}^{\infty} (-1)^m \Big[& H_0^{(1)}\Big(\kappa_0\sqrt{(x_1-x_0-2\ell W)^2+(x_2+y_0-2mH)^2}\Big) \\ & + H_0^{(1)}\Big(\kappa_0\sqrt{(x_1+x_0-2\ell W)^2+(x_2+y_0-2mH)^2}\Big) \\ & + H_0^{(1)}\Big(\kappa_0\sqrt{(x_1-x_0-2\ell W)^2+(x_2-y_0-2mH)^2}\Big) \\ & + H_0^{(1)}\Big(\kappa_0\sqrt{(x_1+x_0-2\ell W)^2+(x_2-y_0-2mH)^2}\Big)\Big]\,, \end{aligned}$$

where $H_0^{(1)}(z)$ denotes the Hankel's function of the first kind.

Modified Biot's Model

The Biot's model treats a poro-elastic medium as an elastic frame with interspinal pore fluid. To formulate a well-posed boundary value problem, we modify the Biot equations in terms of $\mathbf{u}$ and s,

$$\nabla^2 s + \frac{p_{22}}{R} s + \left(p_{12} - \frac{p_{22} Q}{R}\right) e = 0\,, \qquad (3)$$

$$\mu\,\nabla^2\mathbf{u} + \nabla\left[\left(\lambda + \mu - \frac{Q^2}{R}\right) e + \left(\frac{Q}{R} - \frac{p_{12}}{p_{22}}\right) s\right] + \left(p_{11} - \frac{p_{12}^2}{p_{22}}\right)\mathbf{u} = \mathbf{0}\,. \qquad (4)$$

where $\mathbf{u} = (u_1, u_2)$ is the motion of the frame of the bone and $s = Qe + R\epsilon$ is a combination of the solid and fluid dilatations.

Transmission Conditions

Via the Green's representation of P in Ω^w, the solution of (1) can be written in the form of a single-layer potential for the unknown density function φ,

$$P(\mathbf{x},\mathbf{x}_0) = -q\,G(\mathbf{x},\mathbf{x}_0;\kappa_0) - \int_{\partial\Omega^b} G(\mathbf{x},\zeta;\kappa_0)\varphi(\mathbf{x}_0,\zeta)\,dS_\zeta\,, \quad \mathbf{x} \in \Omega^w\,, \qquad (5)$$

Then along the interface between bone and water, we have

- Continuity of the aggregate pressure:

$$\lambda\,\nabla\cdot\mathbf{u} + 2\mu\frac{\partial u_1}{\partial x_1} + Q\,\varepsilon + s = q\,G(\mathbf{X},\mathbf{x}_0;\kappa_0) + \int_{\partial\Omega^b} G(\mathbf{X},\zeta;\kappa_0)\varphi(\mathbf{x}_0,\zeta)\,dS_\zeta \qquad (6)$$

and

$$\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = 0\,. \qquad (7)$$

- Continuity of the flux:

$$\begin{aligned} & \rho^w\omega^2\left(\left[1 - \beta\left(1 + \frac{p_{12}}{p_{22}}\right)\right]\mathbf{n}\cdot\mathbf{u} - \frac{\beta}{p_{22}}\frac{\partial s}{\partial n}\right) + q\frac{\partial G}{\partial n}(\mathbf{X},\mathbf{x}_0;\kappa_0) \\ & = \frac{1}{2}\varphi(\mathbf{x}_0,\mathbf{X}) - \int_{\partial\Omega^b}\varphi(\mathbf{x}_0,\zeta)\frac{\partial G}{\partial n}(\mathbf{X},\zeta;\kappa_0)\,dS_\zeta\,. \end{aligned} \qquad (8)$$

- Continuity of the pore pressure:

$$\beta\int_{\partial\Omega^b} G(\mathbf{X},\zeta;\kappa_0)\varphi(\mathbf{x}_0,\zeta)\,dS_\zeta - s + \beta q\,G(\mathbf{X},\mathbf{x}_0;\kappa_0) = 0\,, \qquad (9)$$

In deriving the equations (6) and (7), we tacitly employed the condition that the tangential frame stress vanishes at the interface.

Numerical Approximation

Because of this reduction, now we only need to discretize the small bone sample domain Ω^b. A second order finite-difference scheme is used to solve the coupled system of equations (3)–(4) and (6)–(9) for the unknowns φ, s, u_1, and u_2.

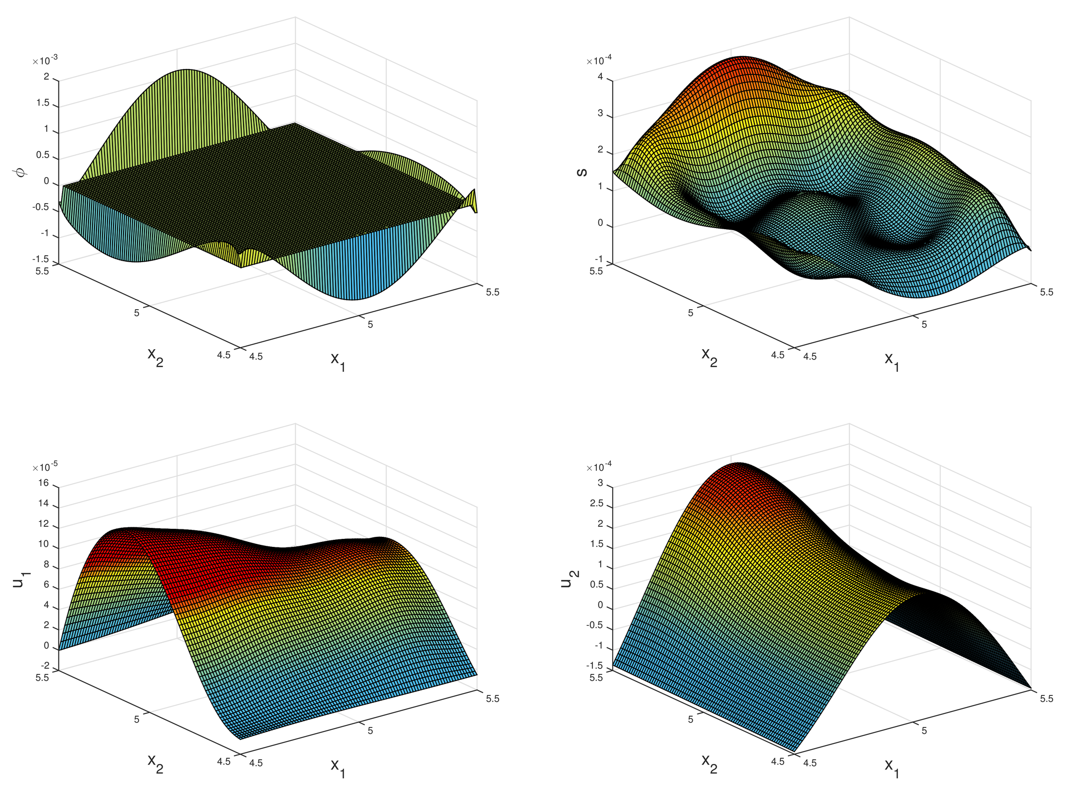

Figure 2: Profiles of φ, s, u_1 and u_2 for $\beta = 0.83$, $\omega = 250$ kHz and $N = 90$.

Recovery of Parameters

The objective function that we use for the sensitivity and recovery tests is the relative root-mean-square error between a reference pressure at each receiving point and the corresponding trial value of P computed by the model.

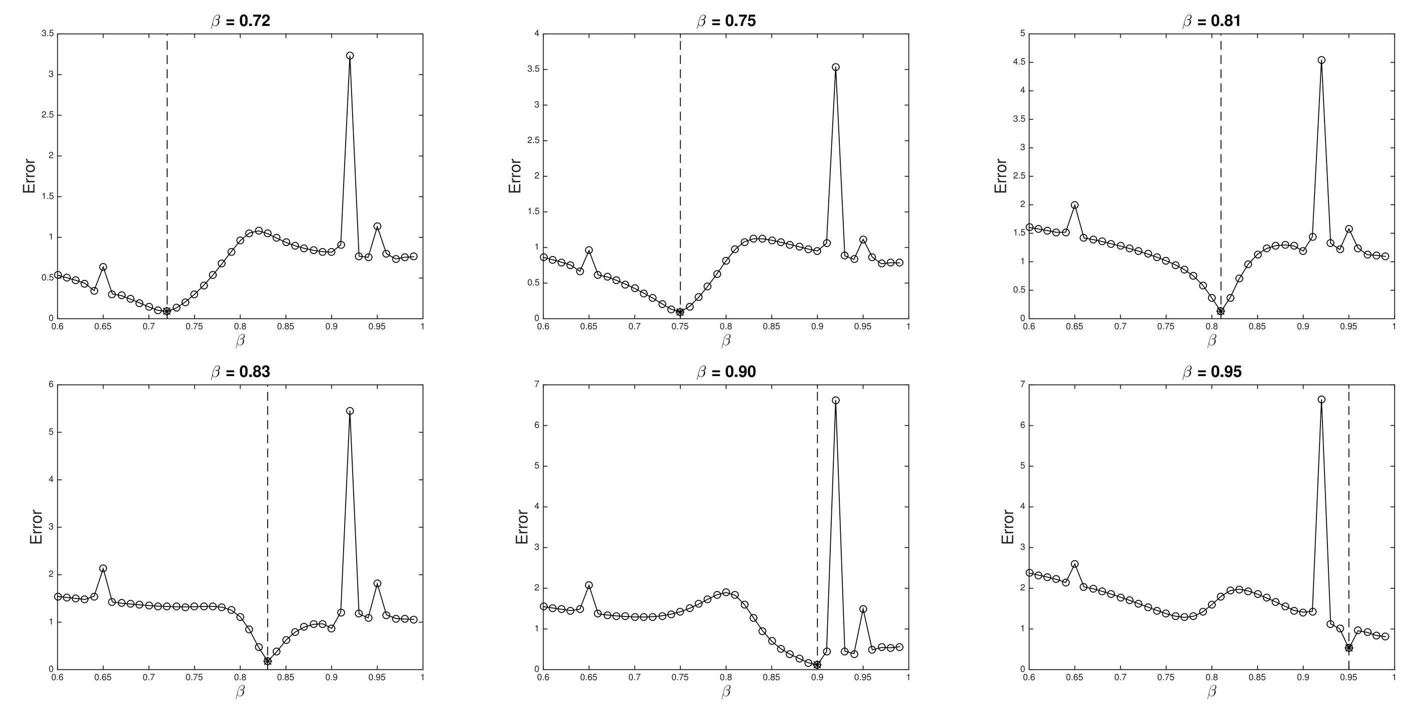

Figure 3: Sensitivity tests on β at $\omega = 250$ kHz with $N = 65$ and 90 for the low and high resolutions respectively.

β target	N low/high	β converged	$f_{\min}$	error %
0.72	25/90	0.7014	0.3247	2.5879
	45/90	0.7128	0.1849	1.0059
	65/90	0.7174	0.0887	0.3613
0.75	25/90	0.7357	0.3627	1.9043
	45/90	0.7452	0.2036	0.6348
	65/90	0.7484	0.0960	0.2148
0.81	25/90	0.7980	0.9406	1.4844
	45/90	0.8109	0.3507	0.1074
	65/90	0.8106	0.1242	0.0781
0.83	25/90	0.8888	0.9747	7.0898
	45/90	0.8322	0.4804	0.2637
	65/90	0.8306	0.1787	0.0684
0.90	25/90	0.9086	0.3982	0.9570
	45/90	0.8949	0.2092	0.5664
	65/90	0.8981	0.0828	0.2148
0.95	25/90	0.9725	0.7834	2.3730
	45/90	0.9482	0.0749	0.1880
	65/90	0.9472	0.3623	0.2917

Table 1: Errors on the recovery of β at $\omega = 250$ kHz for varying resolutions of the trial solution.

Similar results are also obtained for the higher frequency $\omega = 500$ kHz.

Conclusion

- The sensitivity test validates the robustness of the proposed model;
- Based on the univariate minimization, the bone porosity β can be determined to within 2% in most cases and within 0.07% in the best case;
- These results may be viewed as successful and support potential use of this model as a theoretical basis in the development of acoustic techniques for assessing bone strength.