Scattering of electromagnetic waves by delaminated interfaces

Irene de Teresa (in collaboration with **F. Cakoni, C. Lackner and P. Monk**)
University of Delaware, Department of Mathematical Sciences, USA
irene@udel.edu — +1 (302) 831-7381

Motivation

Delamination is a defect that occurs when two materials that should be in contact patially separate. It is important in many engineering systems, and here we present a new model for electromagnetic wave scattering in delaminated configurations, whose advantage is to avoid the numerically expensive process of meshing thin domains. The results will later be used to develop a non-destructive testing for the identification of delamination.

The problem

We are interested in the scattering of an electromagnetic wave by a layered isotropic penetrable obstacle, $\Omega := interior(\overline{\Omega_0 \cup \Omega_+ \cup \Omega_-}) \subset \mathbb{R}^3$, shown in *Figure 1*, Panel (a).

In one part of their interface, the two layers Ω_- and Ω_+ are separated the thin third domain Ω_0, called the **delamination**. Let Γ be the interface between Ω_- and Ω_+ and $\Gamma_0 \subset \Omega_0$ is such that $\overline{\Gamma \cup \Gamma_0}$ is a $\mathcal{C}^2-$ regular surface. The four different domains Ω_0, Ω_-, Ω_+, and $\Omega_{ext} := \mathbb{R}^3 \setminus \overline{\Omega}$ have different continuous physical properties: the relative magnetic permeability μ and the relative electric permitivity ϵ, which are scalar fields. We will assume that in the thin domain Ω_0, the material properties $\mu = \mu_\delta$ and $\epsilon = \epsilon_\delta$ are constant and will denote by $\boldsymbol{\nu}$ the unit normal vector pointing *outwards*.

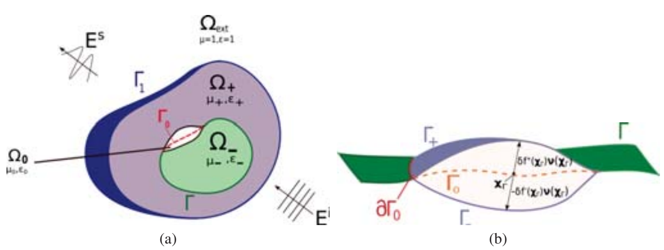

Figure 1: Panel (a) Cross section of the obstacle. The thin layer Ω_0 represents the delamination. Panel (b) Zoom on the delamination.

The well-known equations that model the scattering of the total electromagnetic fields in the frequency domain give rise to:

The standard model

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) - k^2\epsilon\mathbf{E} = 0 \quad \text{in} \quad \Omega_0 \cup \Omega_+ \cup \Omega_- \cup \Omega_{ext}$$

$$\boldsymbol{\nu} \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) \text{ and } \boldsymbol{\nu} \times \mathbf{E} \text{ are continuous in } \mathbb{R}^3,$$

and where, in Ω_{ext}, the electromagnetic field is written as $\mathbf{E} = \mathbf{E}^s + \mathbf{E}^i$, where $\mathbf{E}^i$ denotes the incident field, and $\mathbf{E}^s$ is the radiating field that satisfies the Silver-Müller radiation condition:

$$\lim_{r\to\infty} ((\nabla \times \mathbf{E}^s) \times \hat{\mathbf{x}} - ikr\mathbf{E}^s) = 0, \tag{1}$$

where $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$ and $r = |\mathbf{x}|$.

Preliminaries for a new model

1. Given a parametrization $\mathbf{x}_\Gamma$ of Γ_0, then there is $0 < \eta^*$ such that the map:

$$(\mathbf{x}_\Gamma, \eta) \mapsto \mathbf{x}_\Gamma + \eta\boldsymbol{\nu}(\mathbf{x}_\Gamma), \text{ for all } -\eta^* \leq \eta \leq \eta^*,$$

is an isomorphism in a neighborhood $\mathcal{N}$ of Γ_0.
2. The boundary of the delamination, $\partial\Omega_0$, can be split into $\partial\Omega_0 \cap \Omega_+ = \Gamma_+$ and $\partial\Omega_0 \cap \Omega_- = \Gamma_-$. If the thickness of Ω_0 is small enough, then they can be written in the curvilinear coordinates as:

$$\Gamma^\pm := \{\mathbf{x}_\Gamma \pm \delta f^\pm(\mathbf{x}_\Gamma)\boldsymbol{\nu}(\mathbf{x}_\Gamma) \,:\, \mathbf{x}_\Gamma \in \Gamma_0\},$$

where $0 < \delta < \eta^*$ characterizes the thickness of Ω_0, and $f^\pm : \Gamma_0 \to [0, 1/2]$ are the two functions that describe the profile of the delamination, as shown in *Figure* 1, Panel (b).
3. Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ be an orthonormal set of tangential vectors to the surface Γ_0. Given a smooth vector field $\mathbf{v} = v_1\boldsymbol{\tau}_1 + v_2\boldsymbol{\tau}_2 + v_3\boldsymbol{\nu} \in (\mathcal{C}^\infty(\Gamma_0))^3$ and a smooth scalar field $\rho \in \mathcal{C}^\infty(\Gamma_0)$, we define

$$\mathbf{v}_T = v_1\boldsymbol{\tau}_1 + v_2\boldsymbol{\tau}_2,\; curl_\Gamma\mathbf{v}_T := \partial_1 v_2 - \partial_2 v_1,\; \overrightarrow{curl}_\Gamma\rho := \partial_2\rho\boldsymbol{\tau}_1 - \partial_1\rho\boldsymbol{\tau}_2,\; div_\Gamma\mathbf{v}_T := \partial_1 v_1 + \partial_2 v_2.$$

The asymptotic setting

If the parameter δ we formally assume that the following asymptotic expansions of the fields are valid in a neighborhood of Ω_0:

$$\mathbf{E}^\pm(\mathbf{x}_\Gamma, \eta) = \sum_{l=0}^{\infty} \delta^l \mathbf{E}_l^\pm(\mathbf{x}_\Gamma, \eta) \quad \text{in } \Omega^\pm \text{ and } \mathbf{E}^\delta(\mathbf{x}_\Gamma, \frac{\eta}{\delta}) = \sum_{l=0}^{\infty} \delta^l \mathbf{E}_l(\mathbf{x}_\Gamma, \frac{\eta}{\delta}) \quad \text{in } \Omega_0,$$

where each of the terms $\mathbf{E}_l^\pm(\mathbf{x}_\Gamma, \eta)$, and $\mathbf{E}_l$, $l \geq 0$, are independ of δ for all $l \geq 0$.

The new model

Let the jump and mean values between the surfaces $\Gamma_\pm$ be respectively denoted by:

$$[u] = u^+|_{\Gamma_+} - u^-|_{\Gamma_-} \text{ and } \langle u \rangle = \frac{1}{2}(u^+|_{\Gamma_+} + u^-|_{\Gamma_-}),$$

Using this notation, after neglecting terms of order $O(\delta^2)$, it can be derived:

The new model

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) - k^2\epsilon\mathbf{E} = 0 \quad \text{in} \quad \Omega_+ \cup \Omega_- \cup \Omega_{ext},$$

$$[\boldsymbol{\nu} \times \mathbf{E}] = \mathcal{A}_1\left(\left\langle \frac{1}{\mu}\nabla \times \mathbf{E} \right\rangle_T\right),\; \left[\boldsymbol{\nu} \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right)\right] = \mathcal{A}_2(\langle \mathbf{E}_T \rangle) \text{ on } \Gamma_0,$$

$$\text{and } \boldsymbol{\nu} \times \mathbf{E} \text{ and } \boldsymbol{\nu} \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) \text{ are continuous on } \Gamma_1 \cup \Gamma,$$

where in Ω_{ext}, $\mathbf{E} = \mathbf{E}^s + \mathbf{E}^i$ is the total field, $\mathbf{E}^i$ is an incident field, and $\mathbf{E}^s$ satisfies the Silver-Müller radiation condition (1).

Here the operators $\mathcal{A}_i : \mathcal{H} \to \mathcal{H}^*$ are

$$\mathcal{A}_i(\boldsymbol{\lambda}) = \delta \langle f \rangle \alpha_i \boldsymbol{\lambda} - \delta\overrightarrow{curl}_\Gamma(\langle f \rangle \beta_i curl_\Gamma \boldsymbol{\lambda}),$$

for $i = 1, 2$, where $\langle f \rangle = \frac{f^+ + f^-}{2}$, $\alpha_1 = 2\mu_\delta$, $\alpha_2 = 2k^2\epsilon_\delta$, $\beta_1 = \frac{2}{k^2\epsilon_\delta}$, and $\beta_2 = \frac{2}{\mu_\delta}$,and where

$$\mathcal{H} := \{\mathbf{u} \,|\, \mathbf{u} \in \mathbf{H}^{-1/2}(\Gamma_0),\; curl_\Gamma\mathbf{u} \in H^{-1/2}(\Gamma_0),\; \sqrt{\langle f \rangle}\mathbf{u} \in \mathbf{L}_t^2(\Gamma_0),\; \sqrt{\langle f \rangle}curl_\Gamma\mathbf{u} \in L^2(\Gamma_0)\},$$

$$\|\mathbf{u}\|_{\mathcal{H}}^2 := \|\mathbf{u}\|_{\mathbf{H}^{-1/2}}^2 + \|curl_\Gamma\mathbf{u}\|_{H^{-1/2}}^2 + \left\|\sqrt{\langle f \rangle}\mathbf{u}\right\|_{\mathbf{L}_t^2}^2 + \left\|\sqrt{\langle f \rangle}curl_\Gamma\mathbf{u}\right\|_{L^2}^2.$$

Remark. *Our new model does not include a differential equation in the thin domain Ω_0.*

The variational formulation of the new model

Assumption 1. *Assume that the relative boundary $\partial\Gamma_0$ of Γ_0 in Γ is a $\mathcal{C}^2-$ regular and non self-intersecting curve in $\mathbb{R}^3$, and that:*

a) $f^\pm(\mathbf{x}_\Gamma) = 0$ if and only if $\mathbf{x}_\Gamma \in \partial\Gamma_0$

b) The weight $\langle f \rangle$ satisfies that there is a constant $0 < s < 1$ such that $\lim_{\rho\to 0^+} \frac{\langle f \rangle(\mathbf{x}_\Gamma)}{\rho(\mathbf{x}_\Gamma)^s} = C \neq 0$ for some constant $C \in \mathbb{R}$, where $\rho(\mathbf{x}_\Gamma) = dist(\mathbf{x}_\Gamma, \partial\Gamma_0)$.

Lemma. *Under the hypothesis of Assumption 1, the operator $\mathcal{A}_1 : \mathcal{H} \to \mathcal{H}^*$ is invertible.*

Let B_R be a ball such that $\overline{\Omega} \subset B_R$, and let S_R be the boundary of B_R. Let $H^{-1/2}(div_{S_R}, S_R) := \{\mathbf{u} \,|\, \mathbf{u} \in \mathbf{H}^{-1/2}(S_R), div_{S_R}\mathbf{u} \in H^{-1/2}(S_R)\}$.
Denote by $G_e : H^{-1/2}(div_{S_R}, S_R) \to H^{-1/2}(div_{S_R}, S_R)$ is the well-known exterior electric-to-magnetic Calderón operator defined by $G_e(\boldsymbol{\lambda}) = \hat{\mathbf{x}} \times \mathbf{H}^s$, where $(\mathbf{E}^s, \mathbf{H}^s)$ satisfy

$$ik\mathbf{E}^s + \nabla \times \mathbf{H}^s = 0, \quad ik\mathbf{H}^s - \nabla \times \mathbf{E}^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_R,$$
$$\hat{\mathbf{x}} \times \mathbf{E}^s = \boldsymbol{\lambda} \quad \text{on } S_R, \quad \lim_{r\to\infty} r(\mathbf{H}^s \times \hat{\mathbf{x}} - \mathbf{E}^s) = 0.$$

A variational formulation of the new model is

$$a^+(\mathbf{E}, \mathbf{v}) + b(\mathbf{E}, \mathbf{v}) + ik\int_{S_R} G_e(\hat{\mathbf{x}} \times \mathbf{E}) \cdot \overline{\mathbf{v}}_T \, ds = \mathcal{L}(\mathbf{v}) \text{ for all } \mathbf{v} \in X \tag{2}$$

with

$$a^+(\mathbf{E}, \mathbf{v}) := \int_{B_R^\delta} \left(\frac{1}{\mu}\nabla \times \mathbf{E} \cdot \overline{\nabla \times \mathbf{v}}\right) dV + \int_{\Gamma_0} \langle f \rangle \beta_2 curl_\Gamma \langle \mathbf{E}_T \rangle \, \overline{curl_\Gamma \langle \mathbf{v}_T \rangle} \, ds$$

$$b(\mathbf{E}, \mathbf{v}) := -\int_{B_R^\delta} k^2\epsilon\mathbf{E} \cdot \overline{\mathbf{v}} \, dV - \int_{\Gamma_0} \delta \langle f \rangle \alpha_2 \langle \mathbf{E}_T \rangle \cdot \langle \overline{\mathbf{v}} \rangle_T \, ds + \frac{1}{\delta}\int_{\Gamma_0} \mathcal{A}_1^{-1}([\boldsymbol{\nu} \times \mathbf{E}]) \cdot \overline{[\boldsymbol{\nu} \times \mathbf{v}]} \, ds,$$

$$\mathcal{L}(\mathbf{v}) = \int_{S_R} (\hat{\mathbf{x}} \times (\nabla \times \mathbf{E}^i)) \cdot \overline{\mathbf{v}} - ik \langle G_e(\hat{\mathbf{x}} \times \mathbf{E}_i), \mathbf{v}_T \rangle_{S_R},$$

and the solutions space is

$$X := \{\mathbf{u} \in \mathbf{H}(curl, B_R^\delta) \,|\, \sqrt{\langle f \rangle}\langle \mathbf{u}_T \rangle \in \mathbf{L}_t^2(\Gamma_0)\sqrt{\langle f \rangle}curl_\Gamma \langle \mathbf{u}_T \rangle \in L^2(\Gamma_0)\} \quad \text{with}$$
$$||\mathbf{u}||_X^2 := ||\mathbf{u}||_{H(curl, B_R^\delta)}^2 + ||\sqrt{\langle f \rangle}\langle \mathbf{u}_T \rangle||_{\mathbf{L}_t^2(\Gamma_0)}^2 + ||\sqrt{\langle f \rangle}curl_\Gamma \langle \mathbf{u}_T \rangle||_{L^2(\Gamma_0)}^2.$$

Well-posedness

Theorem. *If, in addition to the hypothesis specified in Assumption 1, there is a constant $c > 0$, such that $\Re(\epsilon^\pm), \Re(\epsilon_\delta), \mu^\pm, \mu_\delta \geq c > 0$, $\Im(\epsilon^\pm) \geq c + \Re(\epsilon^\pm) > 0$ and $\Im(\epsilon_\delta) \geq c + \Re(\epsilon_\delta) > 0$, then the problem 2 has a unique solution for all $\mathcal{L} \in X^*$.*

Numerical validation of the new model

To validate our new model, we present an error analysis based in the comparison between the standard model and our new model. The numerical experiments were implemented in the finite element package Netgen/Ngsolve, in the setting shown in *Figure 2*, Panel (a), with: $r_1 = 1.3$, $r = 0.8$, $f^- = 0$, $f^+ = 1$, $\mu_+ = 1$, $\mu_- = 1$, $\mu_\delta = 2$, $\epsilon_+ = 1. + 1.01i$, $\epsilon_- = 2. + 2.01i$, $\epsilon_\delta = 3.5 + 3.51i$.As incident field, the plane wave $\mathbf{E}^i = pe^{ikd \cdot x}$, with $k = 3$, $d = (0, 0, 1)$, and $p = (1, 0, 0)$.

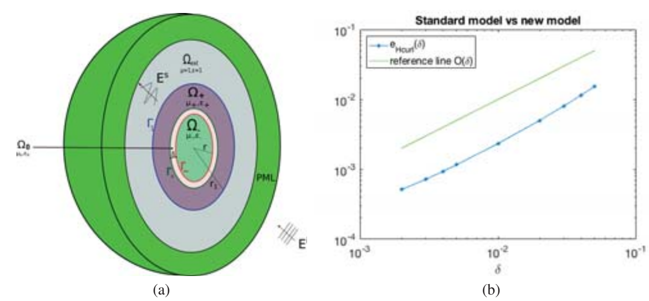

Figure 2: Panel (a) Configuration of the numerical experiments. Panel (b) $H(curl)$ relative error of the total fields resulting from different values of δ. The approximate rate of convergence is $O(\delta)$.

where we defined the relative $H(curl)$ error for each δ by $e_{Hcurl}(\delta) = \frac{||\mathbf{E}_{ATC} - \mathbf{E}_{layer}||_{H(curl, \Omega_{ext})}}{||\mathbf{E}_{layer}||_{H(curl, \Omega_{ext})}}$.

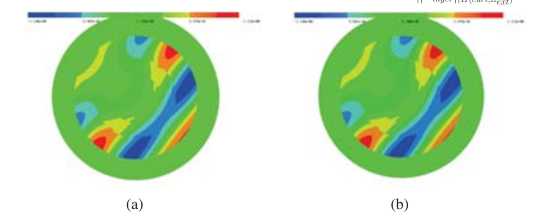

Figure 3: Panel (a) and (b) show respectively the solutions to the standard and new model, for $\delta = 0.003$.

- Therefore, as shown in *Figure* 2, Panel (b), the new model was successfully validated as an $O(\delta)$ approximation of the standard model.