Uniqueness for an Inverse Problem with Formally Determined Offset Data

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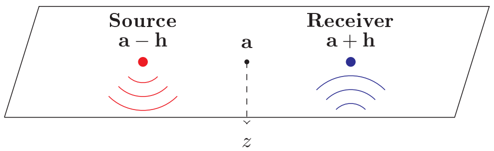
Motivation

The problem under consideration arises in the study of geophysics and medical imaging. Let $\mathbf{x} = (x_1, x_2, z)$ be a point in $\mathbb{R}^3$, $\mathbf{a} = (a_1, a_2, 0)$ and $\mathbf{h} = (h, 0, 0)$ for some $h \geq 0$. We consider an acoustic medium occupying the half space $z \leq 0$ and let $q(\mathbf{x})$ represent some acoustic property of the medium (e.g. oil, minerals, a tumor).



The medium is probed by an acoustic wave, generated at $\mathbf{a} - \mathbf{h}$, and the medium response, $U^{\mathbf{a}}$ is measured at the offset boundary location $\mathbf{a} + \mathbf{h}$, for every $\mathbf{a}$ on the boundary $z = 0$. The goal is to recover the acoustic property $q(\mathbf{x})$ given $U^{\mathbf{a}}(\mathbf{a} + \mathbf{h}, t)$ for all $\mathbf{a}$ on $z = 0$ and for all time t.

Progressing Wave Expansion

Given a real valued $q = q(\mathbf{x})$ on $z \leq 0$ representing the acoustic property of the medium, emit a acoustic wave at $\mathbf{x} = \mathbf{a} - \mathbf{h}$, characterized by $U^{\mathbf{a}}(\mathbf{x}, t)$. It is known that $U^{\mathbf{a}}$ satisfies the following PDE:

$$U^{\mathbf{a}}_{tt} - \Delta_{\mathbf{x}} U^{\mathbf{a}} - qU^{\mathbf{a}} = 0, \qquad \mathbf{x} \in \mathbb{R}^3,\ z \leq 0,\ t \in \mathbb{R} \qquad (1)$$

$$\partial_z U^{\mathbf{a}}(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{a} + \mathbf{h}, t), \qquad \{z = 0\},\ t \in \mathbb{R} \qquad (2)$$

$$U^{\mathbf{a}}(\mathbf{x}, t) = 0, \qquad \mathbf{x} \in \mathbb{R}^3,\ z \leq 0,\ t < 0 \qquad (3)$$

We extend $U^{\mathbf{a}}$ to an even function in z, called $V^{\mathbf{a}}$. Then

$$V^{\mathbf{a}}_{tt} - \Delta_{\mathbf{x}} V^{\mathbf{a}} - qV^{\mathbf{a}} = \delta(\mathbf{x} - \mathbf{a} + \mathbf{h}, t), \qquad \mathbf{x} \in \mathbb{R}^3,\ t \in \mathbb{R}. \qquad (4)$$

$$V^{\mathbf{a}}(\mathbf{x}, t) = 0, \qquad \mathbf{x} \in \mathbb{R}^3,\ t < 0. \qquad (5)$$

Using the "progressing wave expansion" technique, we get

$$V^{\mathbf{a}}(\mathbf{x}, t) = \frac{1}{4\pi} \frac{\delta(t - |\mathbf{x} - \mathbf{a} + \mathbf{h}|)}{|\mathbf{x} - \mathbf{a} + \mathbf{h}|} + v^{\mathbf{a}}(\mathbf{x}, t),$$

where $v^{\mathbf{a}}(\mathbf{x}, t) = 0$ outside of the cone $t = |\mathbf{x} - \mathbf{a} + \mathbf{h}|$ and inside it is the solution of the Goursat problem:

$$v^{\mathbf{a}}_{tt} - \Delta_{\mathbf{x}} v^{\mathbf{a}} - qv^{\mathbf{a}} = 0, \qquad \mathbf{x} \in \mathbb{R}^3,\ t \geq |\mathbf{x} - \mathbf{a} + \mathbf{h}| \qquad (6)$$

$$v^{\mathbf{a}}(\mathbf{x}, |\mathbf{x} - \mathbf{a} + \mathbf{h}|) = \frac{1}{8\pi} \int_0^1 q(\mathbf{a} - \mathbf{h} + s(\mathbf{x} - \mathbf{a} + \mathbf{h}))\, ds, \qquad \mathbf{x} \in \mathbb{R}^3 \qquad (7)$$

Forward Problem

Forward Problem: Given $q(\mathbf{x})$, determine $v^a(\mathbf{x}, t)$.

Pitfall: Cannot measure q directly. The physical measurements we can make are

$$v^{\mathbf{a}}(\mathbf{a} + \mathbf{h}, t), \qquad \mathbf{a} \in \{z = 0\},\ t \in \mathbb{R}.$$

Inverse Problem

Inverse Problem: Given measured data $v^{\mathbf{a}}(\mathbf{a} + \mathbf{h}, t)$, is the coefficient $q(\mathbf{x})$ unique? I.e. Given two solutions of (6)-(7) $v^{\mathbf{a}}_1$ and $v^{\mathbf{a}}_2$ that yield the same measured data $v^{\mathbf{a}}_1(\mathbf{a} + \mathbf{h}, t) = v^{\mathbf{a}}_2(\mathbf{a} + \mathbf{h}, t)$, are the corresponding coefficients q_1 and q_2 equal as well?

To investigate this, we require the condition in the following theorem:

Theorem 1. *If $v^{\mathbf{a}}_1(\mathbf{a} + \mathbf{h}, t) = v^{\mathbf{a}}_2(\mathbf{a} + \mathbf{h}, t)$ for all $\mathbf{a} \in \{z = 0\}$ and $t \in \mathbb{R}$, then $q_1 = q_2$ provided there is a constant C, independent of z such that*

$$\|\nabla_x (q_1 - q_2)(\cdot, z)\|_{L^2(\mathbb{R}^2)} \leq C \|(q_1 - q_2)(\cdot, z)\|_{L^2(\mathbb{R}^2)}, \quad \forall z \in (0, 1], \qquad (8)$$

where $\nabla_x = e_1 \partial_1 + e_2 \partial_2$, the gradient in the first two coordinates.

Method

Physical Data: $v^{\mathbf{a}}(\mathbf{a} + \mathbf{h}, t)$ for $0 \leq t \leq 2\tau$, and the PDE (6)-(7).

Goal: Show the coefficient $q(\mathbf{x})$ is unique for each $v^{\mathbf{a}}(\mathbf{a} + \mathbf{h}, 2\tau)$, i.e. the map $F : q(\mathbf{x}) \mapsto v^{\mathbf{a}}(\mathbf{a} + \mathbf{h}, 2\tau)$ is injective.

Step 1. Formulate the PDE as a difference of two solutions with the same boundary data, $W^{\mathbf{a}} = V^{\mathbf{a}}_1 - V^{\mathbf{a}}_2$, where $p = q_1 - q_2$. Then derive an identity for $W^{\mathbf{a}}$.

Step 2. Derive an identity for a mean value operator of p, $M(p)$ in terms of p and $\int \nabla_x p$.

Step 3. Use steps 1 and 2 to estimate $M(p)$ and $\int \nabla_x p$ in terms of p and $\int p$, then use Gronwall's to determine $p = 0$, i.e. $q_1 = q_2$.

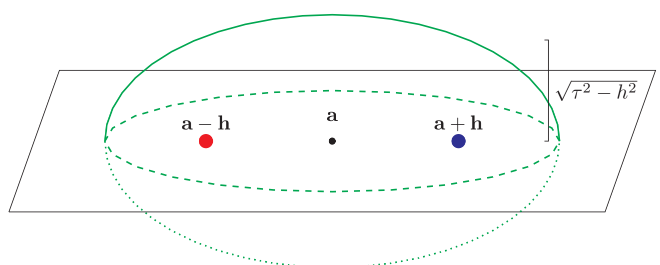
Step 1: The Ellipsoid

Let $W^{\mathbf{a}} = V^{\mathbf{a}}_1 - V^{\mathbf{a}}_2$ where $V^{\mathbf{a}}_1$ and $V^{\mathbf{a}}_2$ solve (4)-(5). Then

$$W^{\mathbf{a}}(\mathbf{a} + \mathbf{h}, 2\tau) = \frac{1}{16\pi^2} \int_{\mathbb{R}^3} p(\mathbf{x}) \delta(\varphi(\mathbf{x}))\, d\mathbf{x} + \iint_{E(\mathbf{a}, \tau)} p(\mathbf{x}) k(\mathbf{x})\, d\mathbf{x}, \qquad (9)$$

where k is smooth and $\varphi(\mathbf{x})$ represents the ellipsoid:

$$\begin{aligned} E(\mathbf{a}, \tau) &= \{0 \leq \varphi(\mathbf{x})\} \\ &= \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - \mathbf{a} + \mathbf{h}| + |\mathbf{x} - \mathbf{a} - \mathbf{h}| \leq 2\tau\} \end{aligned}$$



Step 2: Mean Value Identity

We can rewrite the first integral in (9) as a "mean value" operator,

$$(Mp)(\mathbf{a}, \tau) = \frac{1}{16\pi^2} \int_{\partial E(\mathbf{a}, \tau)} \frac{p(\mathbf{x})}{|\nabla \varphi(\mathbf{x})|}\, dS_{\mathbf{x}}.$$

Taking a derivative in $\sigma = \sqrt{\tau^2 - h^2}$, we can extract the value of p at the north pole of the ellipsoid, $\mathbf{a} + \sigma \mathbf{e}_3$, and get the following estimate

$$|p(\mathbf{a} + \sigma \mathbf{e}_3)|^2 \preccurlyeq \left| \frac{\partial}{\partial \sigma} (Mp)(\mathbf{a}, \tau) \right|^2 + \int_{\partial E(\mathbf{a}, \tau)} \frac{|\nabla_y p(\mathbf{x})|^2}{\sqrt{\sigma^2 - z^2}}\, dS_{\mathbf{x}}. \qquad (10)$$

Step 3: Estimates

Let $W^{\mathbf{a}} = 0$ i.e. $V^{\mathbf{a}}_1 = V^{\mathbf{a}}_2$. We first estimate

$$\left| \frac{\partial}{\partial \sigma} (Mp)(\mathbf{a}, \tau) \right|^2 \preccurlyeq \int_{\partial E(\mathbf{a}, \tau)} |p(\mathbf{x})|^2\, dS_{\mathbf{x}}.$$

Thus from (10), we get

$$|p(\mathbf{a} + \sigma \mathbf{e})|^2 \preccurlyeq \int_{\partial E(\mathbf{a}, \tau)} |p(\mathbf{x})|^2\, dS_{\mathbf{x}} + \int_{\partial E(\mathbf{a}, \tau)} \frac{|\nabla_x p(\mathbf{x})|^2}{\sqrt{\sigma^2 - z^2}}\, dS_{\mathbf{x}} \qquad (11)$$

Let $P(z) = \int_{\mathbb{R}^2} |p(x, z)|^2 dx$, then from the estimates (8) and (11), we get

$$P(\sigma) \leq C \int_0^{\sigma} P(z)\, dz \qquad \forall \sigma \in (0, 1].$$

From Gronwall's inequality, this gives that $P(\sigma) = 0$, implying $p = 0$ i.e. $q_1 = q_2$.

Future Work and References

Future Work: A similar problem, but instead of working over the ellipsoid,

$$|\mathbf{x} - (\mathbf{a} - \mathbf{h})| + |\mathbf{x} - (\mathbf{a} + \mathbf{h})| \leq 2\tau$$

we work over the hyperboloid

$$|\mathbf{x} - \mathbf{a}| - |\mathbf{x}| \leq \tau$$

References:

[1] Rakesh and G. Uhlmann *The Point Source Inverse Back-Scattering Problem*, Contemporary Mathematics 644, 11 pp, (2015).

[2] Rakesh and G. Uhlmann. *Uniqueness for the inverse back-scattering problem for angularly controlled potentials*, Inverse Problems 30, 28 pp, (2014).

[3] V. G. Romanov. *Integral Geometry and Inverse Problems for Hyperbolic Equations*, Springer Tracts in Natural Philosophy, Volume 26, (1974).