# A Random Coefficient PDE Model for Damage Mechanics

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## The Damage Mechanics Problem

We present a damage mechanics model for fracture in brittle materials undergoing uniaxial loading. In a perfectly brittle material, the stress-strain relation behaves linearly until a critical point is reached, at which point the material fractures and the stress returns to zero. To ensure mathematical feasibility and continuity of our model, we modify the stress-strain relation to include a softening region, which extends the time of fracture by introducing material weakening.

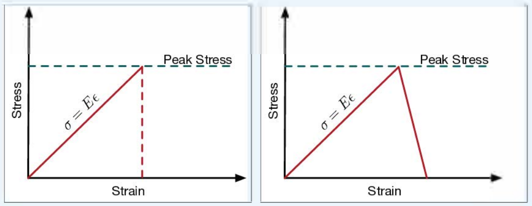


Figure 1. Stress-Strain relation in the damage mechanics problem. The idealized relation (left) shows instantaneous fracture, while the softened relation (right) shows an allowed material weakening region.

### *At a Glance*

- **Goal:** generate a quantity of interest - the stress-strain curve of a material - and to understand the uncertainty in this quantity due to the random distribution of material defects
- **Challenges:** this model is a random coefficient nonlinear PDE, raising questions of well-posedness and computational feasibility
- **Our Solution:** in [2] we prove the random coefficient problem is well posed in the properly chosen function space for almost all of the probability space, and demonstrate computational feasibility using direct Monte-Carlo (MC) sampling methods

## Damage Mechanics Model

Material damage is computed in relation to the strain. Damage is zero until a critical strain value, $\epsilon^*(x)$, is reached. Afterwards damage increases to a value of 1.

$$\tilde{D}(\epsilon) = \begin{cases} 0 & \epsilon < \epsilon^*(x;\omega) \\ \dfrac{\epsilon^*(x;\omega) + \Delta\epsilon}{\Delta\epsilon}\left(1 - \dfrac{\epsilon^*(x;\omega)}{\epsilon}\right) & \epsilon^*(x;\omega) \le \epsilon \le \epsilon^*(x;\omega) + \Delta\epsilon \\ 1 & \epsilon^*(x;\omega) + \Delta\epsilon < \epsilon \end{cases} \quad (1)$$

Here $\omega$ is a hidden random variable which governs the random defects of the material.

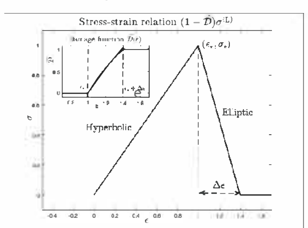


Figure 2. Damage profile for our model. There is no damage until a critical value is reached. At which point the damage is linear in relation to the stress.

To ensure damage is irreversible in time, we define:

$$D(\epsilon(t)) = \sup_{s \le t}\left\{\tilde{D}(\epsilon(t))\right\} \quad (2)$$

## Random Critical Strain

To simulate real materials, our critical strain is distributed according to a log-normal distribution. Samples of $\epsilon^*(x;\omega)$ represent applying the uniaxial loading conditions on various samples of a particular material.

$$\log\left(\epsilon^*(x;\omega)\right) \sim \mathcal{N}(m(x), \mathcal{C}_\ell). \quad (3)$$

The function $m(x)$ is the mean log-critical-strain, and $\mathcal{C}_\ell$ is the covariance operator, chosen to be the solution operator to the problem:

$$\frac{d^2 v}{dx^2} + \frac{v}{\ell^2} = f(x), \quad v(0) = v(1) = 0, \mathcal{C}_\ell : f \mapsto v. \quad (4)$$

This distribution can be sampled using the **Karhunen-Loéve Expansion.**

## One Dimensional Random Coefficient Problem

We approximate the damage mechanics problem in one dimension. Nondimensionalizing all quantities yields the problem formulation:

$$\frac{\partial^2 u}{\partial t^2} + \eta\frac{\partial u}{\partial t} - \nu\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}((1 - D(\epsilon))\epsilon), \quad x \in [0,1], \quad t > 0. \quad (5)$$

By nondimensionalizing, the linear material stress is equal to that of the material strain, and so the two variables $\sigma$ and $\epsilon$ may be used interchangeably. The random yield stress of the material is captured by the damage function $D(\epsilon)$. Since we assume the material is undergoing a constant expansion rate at the right-most end, the boundary and initial conditions are

$$u(x, t = 0) = \frac{du}{dt} = 0, \quad u(x = 0, t) = 0, \quad u(x = 1, t) = ct. \quad (6)$$

### *Proposition*

*The random coefficient nonlinear PDE defined in 5 with damage defined in 1, 2 is well posed [2].*

## Numerical Discretization and Solution

To solve the damage mechanics problem we employ a finite-difference discretization for the problem. The spatial domain is discretized using $M + 2$ equally spaced points:

$$0 = x_0 < x_1 < \dots < x_M < x_{M+1} = 1, \quad u_j(t) \approx u(x_j, t).$$

Using second order centered differences we convert the operators $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial}{\partial x}\left(1 - D\left(\frac{\partial u}{\partial x}\right)\right)\frac{\partial u}{\partial x}$ into the (quasi) linear operators $A_L\boldsymbol{u}$ and $A_N(\boldsymbol{u})\boldsymbol{u}$ respectively. The MOL semidiscretized problem reads

$$\frac{d^2\boldsymbol{u}}{dt^2} + \eta\frac{d\boldsymbol{u}}{dt} - \nu\frac{d}{dt}A_L\boldsymbol{u} - A_N(\boldsymbol{u})\boldsymbol{u} = \boldsymbol{g}(t), \quad u_j(t) \approx u(x_j, t),$$

and $\boldsymbol{g}(t)$ is due to the boundary condition at $x = 1$. Solving the problem forward in time is done using the Newmark-$\beta$ method.

$$\begin{bmatrix} I & 0 & -\beta\Delta t^2 I \\ 0 & I & -\gamma\Delta t I \\ -A_N(\boldsymbol{u}^n) & \eta I - \nu A_L & I \end{bmatrix}\begin{bmatrix} \boldsymbol{u}^{n+1} \\ \boldsymbol{v}^{n+1} \\ \boldsymbol{a}^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}^n + \Delta t\boldsymbol{v}^n + \frac{(1-2\beta)}{2}\Delta t^2\boldsymbol{a}^n \\ \boldsymbol{v}^n + (1-\gamma)\Delta t\boldsymbol{a}^n \\ \boldsymbol{g}^n \end{bmatrix}, \quad (7)$$

Where

$$\boldsymbol{u}^n \approx \boldsymbol{u}(n\Delta t), \quad \boldsymbol{v}^n \approx \frac{d\boldsymbol{u}}{dt}(n\Delta t), \quad \boldsymbol{a}^n \approx \frac{d^2\boldsymbol{u}}{dt^2}(n\Delta t), \quad \boldsymbol{g}^n = g(n\Delta t).$$

Setting time stepping parameters $\beta = 0.25$ and $\gamma = 0.5$, this method is equivalent to midpoint method and is unconditionally stable.

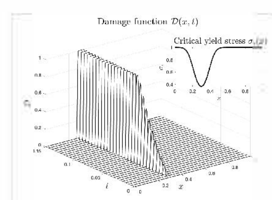


Figure 3. Damage profile in both space and time for a sample problem. The critical stress profile is inset to show damage occurring at the weak point in the material.

## Convergence Analysis

We did a convergence analysis for the 1D problem described above using a fixed critical strain:

$$\epsilon^*(x) = \exp\left(-\exp\left(-100(x - 0.3)^2\right)\right), \quad \Delta\epsilon = 0.1.$$

The expansion speed was fixed at $c = 0.5$ and the simulation time was set to a maximum time of 1. The Rayleigh damping parameters were set to $\eta = \nu = 0.1$.

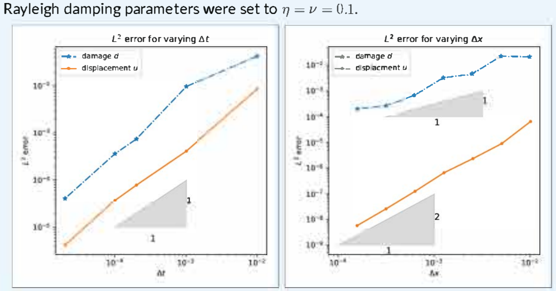


Figure 4. Observed convergence results for numerical discretization scheme. The problem is second order convergent in space as expected, and is first order convergent in time.

## Ensemble Simulations

We solve the damage mechanics problem for a series of 1000 random samples of the random critical yield stress field defined in 3, 4 for various material correlation lengths, $\ell$. In each ensemble of simulations, we plot the total integrated material stress over time. In the below figures we are able to see where the nonlinearity in the model takes effect, which in turn shows when the material begins taking damage.

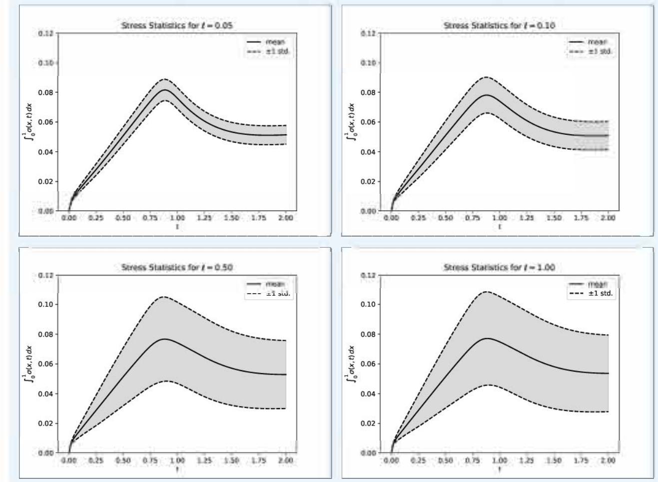


Figure 5. Global mean stress vs. time curves for correlation lengths 0.05 (top left), 0.10 (top right), 0.50 (bottom left), and 1.00 (bottom right) together with 1 standard-deviation uncertainty interval. Since the material underwent a constant expansion rate, the strain is directly proportional to time.

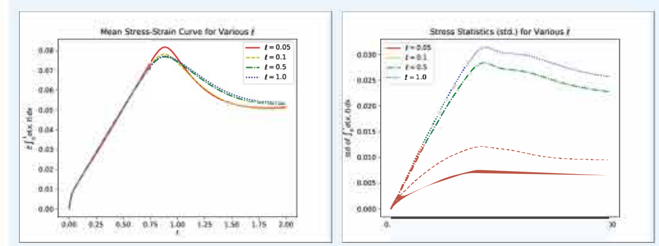


Figure 6. Comparison of stress-strain means (left) and standard deviations (right) over various correlation lengths examined. Since the material underwent a constant expansion rate, the strain is directly proportional to time.

## Conclusions and Outlook

We demonstrated the numerical feasibility of the one-dimensional random yield strength problem for material damage. In [2] we proved the well-posedness of the damage mechanics problem. Furthermore we demonstrated the ability to characterize material stress behavior for various random critical yield strains.

Future work is focused on efficient sampling methods for the random coefficient PDE problem. A Multilevel Monte-Carlo (MLMC) approach can be used to sample from the distribution of random yield strengths for a given material, and gain high accuracy on various observables such as the mean and standard deviation of the stress-strain curves seen above, while minimizing computational costs. This approach can then be applied to higher dimensional problems for damage mechanics which can result in high accuracy and low cost.

## References

[1] Zdenek P. Bažant and Ted B. Belytschko. Wave propagation in a strain-softening bar: exact solution. *J. Eng. Mech.*, 111(3):381–389, 1985.

[2] Petr Plechac, Gideon Simpson, and Jerome Troy. Well-posedness of a random coefficient damage mechanics model. *Applicable Analysis*, 2022.

[3] Roger M. Temam and Alain M. Miranville. *Mathematical Modeling in Continuum Mechanics*, chapter 3. Cambridge University Press, 2005.