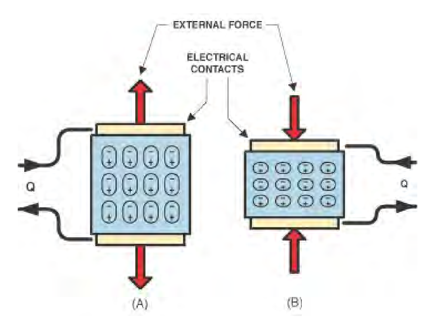
TIME-DOMAIN ACOUSTIC SCATTERING BY PIEZOELECTRIC OBSTACLES

TONATIUH SÁNCHEZ-VIZUET AND FRANCISCO-JAVIER SAYAS {tonatiuh,fjsayas}@udel.edu

WHAT IS PIEZOELECTRICITY?

Piezoelectricity is a physical effect present on some materials that connects their elastic and electric properties. Roughly speaking, on a piezoelectric solid elastic deformations produce electric fields and viceversa.

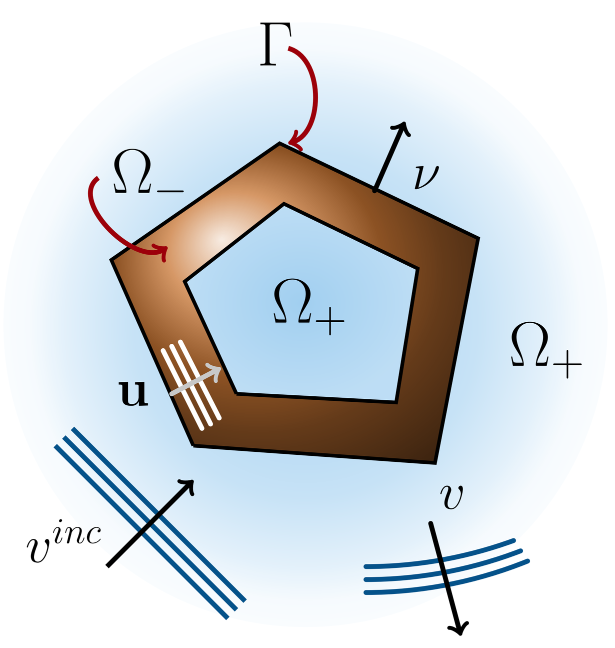


On the linear models of piezoelectricity, the elastic displacement vetor $\mathbf{u}$ and the electric potential ψ are connected through the constitutive relations

$$\boldsymbol{\sigma} := \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{e}\nabla\psi, \qquad \mathbf{D} := \mathbf{e}^\top\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\epsilon}\nabla\psi,$$

that define the elastic stress tensor and the electric displacement vector respectively. In the above definition $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right)$ is the linear elastic strain tensor, $\mathbf{C}$, $\mathbf{e}$, and $\boldsymbol{\epsilon}$ are the elastic compliance, piezoelectric, and dielectric tensors respectively. *Image Source: http://archives.sensorsmag.com/articles/0203/33/main.shtml

THE SCATTERING PROBLEM



We are interested in studying the interaction between an incident acoustic wave v^{inc} and a piezoelectric scatterer. The total acoustic field

$$v^{tot} = v^{inc} + v$$

is the supersposition of the incident wave and an unknown scattered wave v. The following system relates the scattered wave v with the elastic displacement vector $\mathbf{u}$ and the electric potential ψ

$$\begin{aligned}
\Delta v &= c^{-2}v_{tt} && \text{in } \Omega_+ \times [0,\infty),\\
\nabla\cdot\boldsymbol{\sigma} &= \rho_\Sigma \mathbf{u}_{tt} && \text{in } \Omega_- \times [0,\infty),\\
\nabla\cdot\mathbf{D} &= 0 && \text{in } \Omega_- \times [0,\infty),\\
\mathbf{u}_t\cdot\boldsymbol{\nu} + \partial_\nu v &= -\partial_\nu v^{inc} && \text{on } \Gamma \times [0,\infty),\\
\boldsymbol{\sigma}\boldsymbol{\nu} + \rho_f v_t \boldsymbol{\nu} &= -\rho_f \partial_t v^{inc}\boldsymbol{\nu} && \text{on } \Gamma \times [0,\infty),\\
\mathbf{D}\cdot\boldsymbol{\nu} &= \eta_d && \text{on } \Gamma_N \times [0,\infty),\\
\psi &= \mu_d && \text{on } \Gamma_D \times [0,\infty),
\end{aligned}$$

with homogeneous initial conditions for v, v_t, $\mathbf{u}$, and $\mathbf{u}_t$. In order to ensure uniqueness of the solution, we assume that the Dirichlet boundary Γ_D is non-empty.

INTEGRO-DIFFERENTIAL SYSTEM

Let $s \in \mathbf{C}_+$ and define the boundary integral operator

$$\mathbb{D}(s) := \begin{bmatrix} \mathrm{V}(s) & +\frac{1}{2}\mathrm{I} - \mathrm{K}(s) \\ -\frac{1}{2}\mathrm{I} + \mathrm{K}^\top(s) & \mathrm{W}(s) \end{bmatrix}$$

and the piezoelectric bilinear form

$$\begin{aligned}\mathcal{B}((\mathbf{u},\psi),(\mathbf{w},\varphi);s) :=& (\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{w}))_{\Omega_-} + s^2(\rho_\Sigma\mathbf{u},\mathbf{w})_{\Omega_-}\\ &+ (\mathbf{e}\nabla\psi,\boldsymbol{\varepsilon}(\mathbf{w}))_{\Omega_-} - (\boldsymbol{\varepsilon}(\mathbf{u}),\mathbf{e}\nabla\varphi)_{\Omega_-} + (\boldsymbol{\epsilon}\nabla\psi,\nabla\varphi)_{\Omega_-}\end{aligned}$$

In the Laplace-domain, the scattering problem is equivalent to the variational problem of, given data $(\alpha_d,\beta_d,\eta_d,\mu_d)$, finding $(\mathbf{u},\psi,\lambda,\phi)$ such that for all test functions $(\mathbf{w},\varphi,\xi,\chi)$

$$\begin{aligned}
\gamma_D\psi &= \mu_d\\
-s\langle\gamma\mathbf{u}\cdot\boldsymbol{\nu},\chi\rangle_\Gamma + \langle\mathbb{D}(s/c)(\lambda,\phi),(\xi,\chi)\rangle_\Gamma &= \langle\alpha_d,\chi\rangle_\Gamma\\
\mathcal{B}((\mathbf{u},\psi),(\mathbf{w},\varphi);s) + \rho_f s\langle\phi,\gamma\mathbf{w}\cdot\boldsymbol{\nu}\rangle_\Gamma &=\\
-\rho_f s\langle\beta_d,\gamma\mathbf{w}\cdot\boldsymbol{\nu}\rangle_\Gamma &+ \langle\eta_d,\gamma\varphi\rangle_{\Gamma_N}
\end{aligned}$$

With an appropriate choice of function spaces, the above problem can be shown to be uniquely solvable and the stability bounds can be used to establish error bounds for the discrete problem in the time-domain.

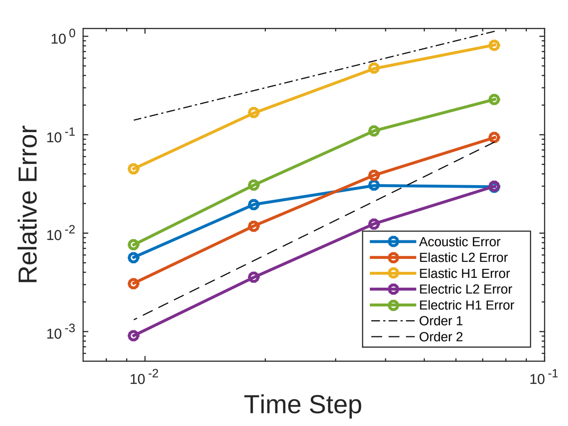
CONVERGENCE STUDIES

Theorem 1. *Let $(\alpha_d,\beta_d,\eta_d,\mu_d^h \equiv \mu_d)$ sufficiently smooth causal problem data. If BDF2 Convolution Quadrature is used for time discretization, for $t \geq 0$, the approximation error satisfies*

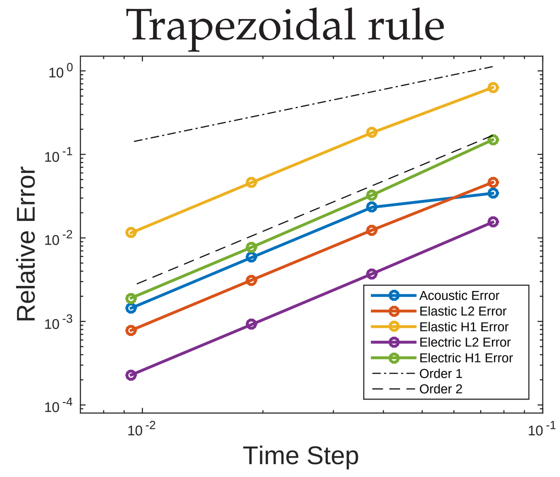
$$\begin{aligned}&\|(v,\mathbf{u},\psi)(t) - (v_\kappa^h,\mathbf{u}_\kappa^h,\psi_\kappa^h)(t)\|_{H_1}\\ &\quad\leq C_1(1+t^2)\kappa^2 + C_2 Approx(h),\end{aligned}$$

where C_1 and C_2 depend only on Γ and $Approx(h)$ gives the history of best spatial approximations to the exact solution.

BDF2



Trapezoidal rule



Spacial discretization was done using $\mathcal{P}_2$ Lagrangian finite elements for the elastic and electric variables and $\mathcal{P}_2/\mathcal{P}_1$ continuous/discontinuous Galerkin boundary elements for the acoustic wavefield. For time evolution we compared BDF2 and trapezoidal rule-based Convolution Quadrature.

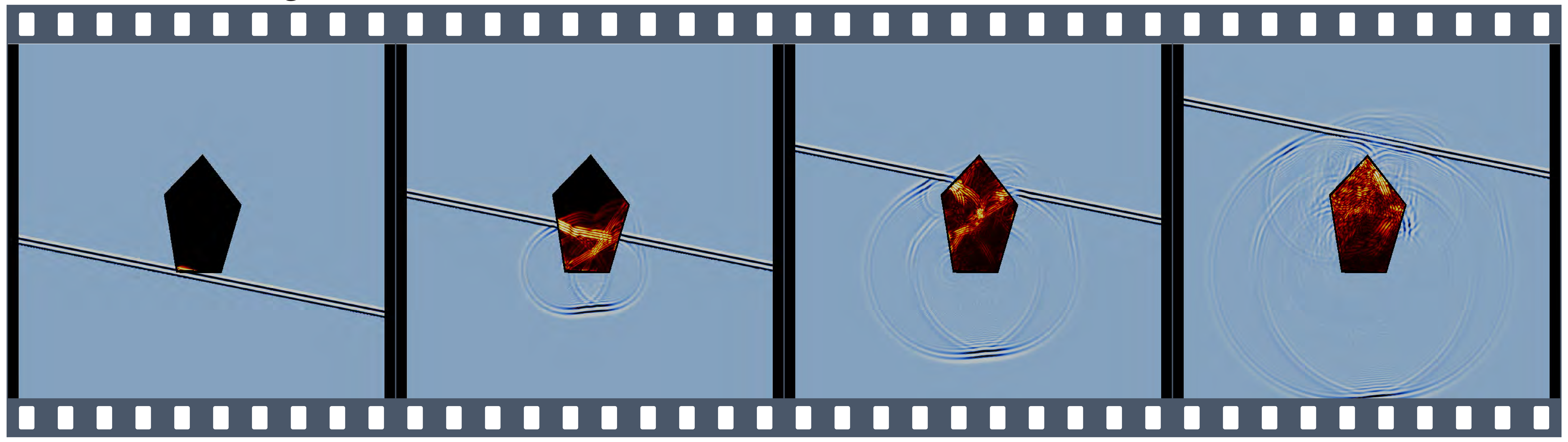
SIMULATIONS

The simulation shows the sinusoidal acoustic planar pulse

$$v^{inc} = 3\chi_{[0,0.3]}(88s)\sin(88s), \qquad s := (t - \mathbf{r}\cdot\mathbf{d})$$
$$\mathbf{r} := (x,y), \qquad \mathbf{d} := (1,5)/\sqrt{26}.$$

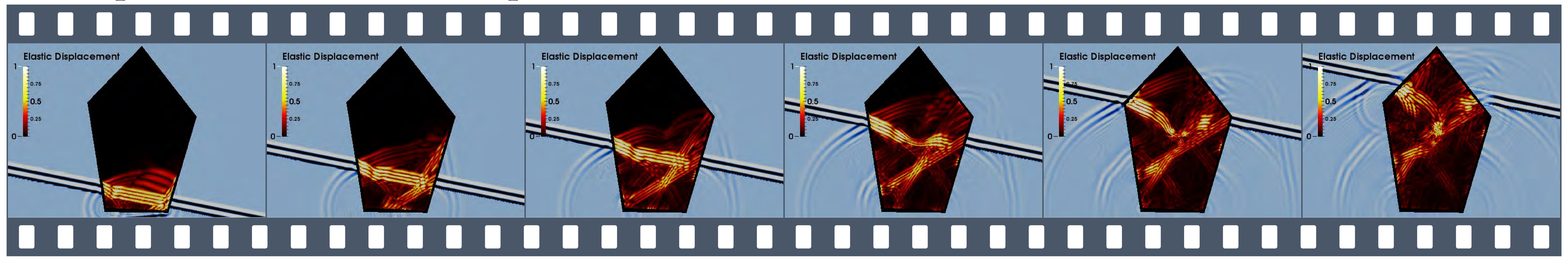
interactig with a piezoelectric scatterer with a gaussian mass distribution. The entirety of the solid/fluid interfase is used as the Dirichlet boundary, where the grounding condition $\psi = 0$ on $\Gamma_D \times [0,\infty)$ was imposed. Here $\mathcal{H}$ is a smooth approximation to the Heaviside step function. The simulation was done with $\mathcal{P}_2$ FEM and $\mathcal{P}_2/\mathcal{P}_1$ BEM coupled with trapezoidal rule-based CQ. The time step was $\kappa = 0.0035$ and the mesh parameter was $h = 0.007$.

Acoustic scattering

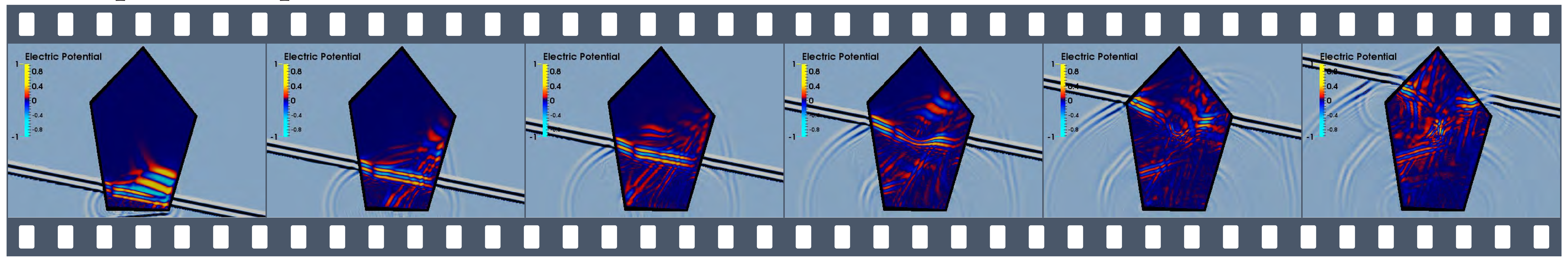


Time 0.175s 0.7s 1.225s 1.75s

Closeup: Norm of the elastic displacement



Closeup: Electric potential



Time 0.35s 0.525s 0.7s 0.875s 1.05s 1.225s

REFERENCES & ACKNOWLEDGEMENTS

The authors were partially supported by the National Science Foundation through the grant DMS 1216356.

- M. Hassell and F-J. Sayas. Convolution Quadrature for wave simulations. *To appear in SEMA-SIMAI Lecture Notes in Mathematics. Springer*, 2016.
- G. Hsiao, T. Sánchez-Vizuet, and F-J. Sayas. Boundary and coupled boundary-finite element methods for transient wave-structure interaction. *Accepted at IMA J. Numer. Anal.*, 2015.
- T. Sánchez-Vizuet and F-J. Sayas. Symmetric BEM-FEM coupling for time-domain acoustic scattering by obstacles with piezoelectric behavior. *In preparation.*