The Factorization Method for a Cavity in an Inhomogeneous Medium

Shixu Meng

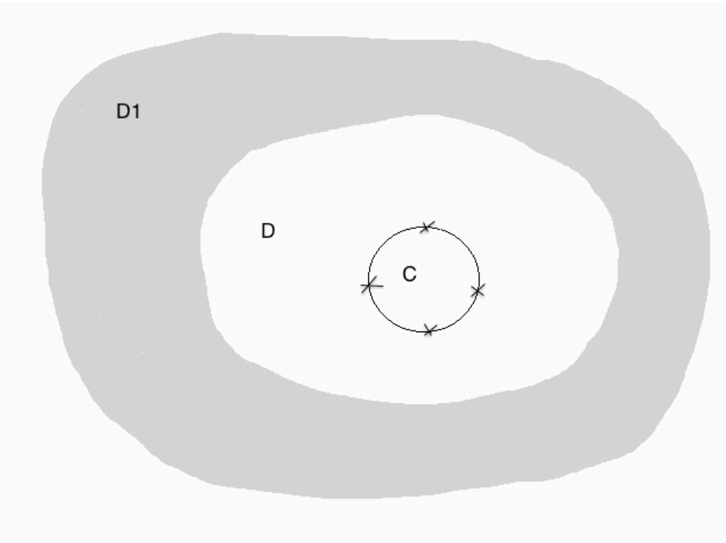
In collaboration with F. Cakoni and H. Haddar

Department of Mathematical Sciences, University of Delaware

ABSTRACT

We consider the inverse scattering problem for a cavity that is bounded by a penetrable anisotropic inhomogeneous medium of compact support and seek to determine the shape of the cavity from internal measurements on a curve or surface inside the cavity. We derive a factorization method which provides a rigorous characterization of the support of the cavity in terms of the range of an operator which is computable from the measured data. The support of the cavity is determined without a-priori knowledge of the constitutive parameters of the surrounding anisotropic medium provided they satisfy appropriate physical as well as mathematical assumptions imposed by our analysis.

PHYSICAL SETTINGS



- The cavity is $D \subset \mathbb{R}^d$, $d = 2, 3$, a simply connected bounded region of $\mathbb{R}^d$ with Lipshitz boundary ∂D.
- The anisotropic inhomogeneous layer is $D_1 \backslash \overline{D}$ in which $A = (a_{ij})_{d\times d}$, $a_{ij} \in L^\infty(D_1\backslash\overline{D})$, $\mathrm{Re}(A)$ positive definite, $\mathrm{Im}(A)$ negative semidefinite, $n \in L^\infty(D_1\backslash\overline{D})$.
- $\overline{C} \subset D$, the sources and receivers are on ∂C.
- Point source located at y is given by

$$\Phi(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & \text{in } \mathbb{R}^2 \\ \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} & \text{in } \mathbb{R}^3. \end{cases}$$

FORWARD PROBLEM

Given point source $\Phi(\cdot,y)$ located at y along the curve ∂C inside the cavity D, consider the scattering of $\Phi(\cdot,y)$ by the inhomogeneous media, the scattered field $u^s(\cdot,y)$ satisfies

$$\nabla\cdot A\nabla u^s(\cdot,y) + k^2 n u^s(\cdot,y) = \nabla(I-A)\nabla\Phi(\cdot,y) + k^2(1-n)\Phi(\cdot,y) \quad \text{in} \quad \mathbb{R}^d$$

and Sommerfeld radiating condition. Using a variational approach (see e.g. [3]), it is well known that the forward scattering problem has a unique solution in $H^1_{loc}(\mathbb{R}^d)$ which depends continuously on $\Phi(\cdot,y)$.

FACTORIZATION METHOD

The data set defines the *near field operator* $N : L^2(\partial C) \to L^2(\partial C)$

$$(Ng)(x) = \int_{\partial C} u^s(x,y)g(y)ds(y)$$

where $g \in L^2(\partial C)$ and $x \in \partial C$.

Define the bounded linear operator $H : L^2(\partial C) \to H^1(D_1\backslash\overline{D})$ by

$$(Hg)(x) := \int_{\partial C} \overline{\Phi(x,y)}g(y)ds(y), \; x \in D_1\backslash\overline{D}$$

Lemma 1 For $z \in D_1\backslash\overline{C}$ we have that

$$\Phi(\cdot,z) \in \mathrm{Range}(H^*) \text{ if and only if } z \in D_1\backslash\overline{D}.$$

where H^* is the adjoint operator of H.

Note: Unfortunately H^* depends on D, therefore it it can not be used to reconstruct ∂D.

The following allows us to determine ∂D by using the near field operator.

Lemma 2 The near field operator can be factorized as

$$N = H^* S H$$

where under certain assumption S is defined via an apropriate transmission boundary value problem and guarantees that

$$\mathrm{Range}(N_\#) = \mathrm{Range}(H^*)$$

where $N_\# := |\mathrm{Re}(N)| + \mathrm{Im}(N)$ and

$$\mathrm{Re}(N) = \frac{N+N^*}{2}, \mathrm{Im}(N) = \frac{N-N^*}{2i}$$

MAIN THEOREM

Assumption 1: The wave number $k > 0$ is such that there does not exist a nonzero $w_0 \in H^1(D_1\backslash\overline{D})$ satisfying for any $\psi \in H^1(D_1\backslash\overline{D})$

$$\int_{D_1\backslash\overline{D}} (I-A)\nabla w_0 \cdot \nabla\overline{\psi} - k^2 \int_{D_1\backslash\overline{D}} (1-n)w_0\overline{\psi} = 0.$$

We denote by $(\phi_j, \lambda_j)_{j\in\mathbb{N}}$ an orthonormal eigensystem for $N_\#$. Then by Picard's theorem we have the following result.

Theorem: Suppose that Assumption 1 is valid for the wave number $k > 0$, and either $\mathrm{Re}(A) > I$, or $I - \mathrm{Re}(A) - \alpha|\mathrm{Im}(A)| > 0$ and $\mathrm{Re}(A) - \frac{1}{\alpha}|\mathrm{Im}(A)| \geq 0$ for some $\alpha > 0$. Then for $z \in D_1\backslash\overline{C}$

$$z \in D_1\backslash\overline{D} \quad \text{if and only if} \quad \sum_j \frac{|(\Phi_z, \phi_j)|^2}{|\lambda_j|} < \infty$$

where $\Phi_z := \Phi(\cdot,z)|_{\partial C}$, with $\Phi(\cdot,z)$ being the fundamental solution of the Helmholtz equation [1].

References

[1] S. Meng, H. Haddar and F. Cakoni, The Factorization Method for a Cavity in an Inhomogeneous Medium, Inverse Problems, accepted.

[2] F. Cakoni, D. Colton, S. Meng, The inverse scattering problem for a penetrable cavity with internal measurements, AMS Contemporary Mathematics, to appear.

[3] F. Cakoni and D. Colton (2014) *A Qualitative Approach to Inverse Scattering Theory* Springer, Berlin.

Acknowledgment: Research supported by NSF Grant DMS-1106972 and DRI - INRIA

INVERSE PROBLEM

We can choose C such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in C. Then we place the point source $\Phi(\cdot,y)$ at every $y \in \partial C$ and measure the corresponding scattered field $u^s(x,y)$ for $x \in \partial C$.

The *inverse problem* is for fixed (but not necessarily known) A and n, determine the boundary of the cavity ∂D from a knowledge of $u^s(x,y)$ for all $x, y \in \partial C$.

Uniqueness: D is uniquely determined from a knowledge of scattered fields on ∂C for all point sources located on ∂C [2].

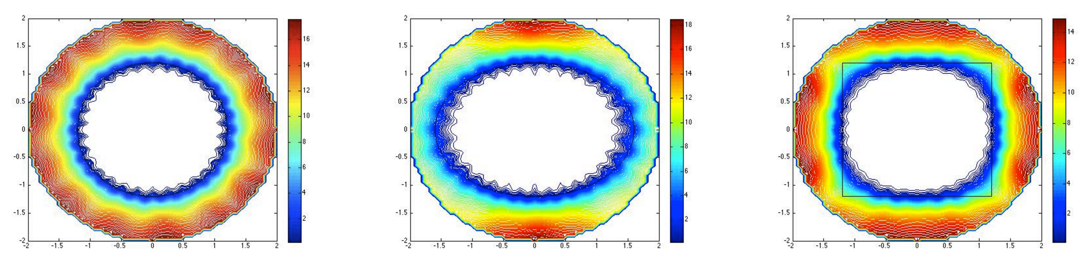
NUMERICAL EXAMPLE

We plot the indicator function of the cavity D

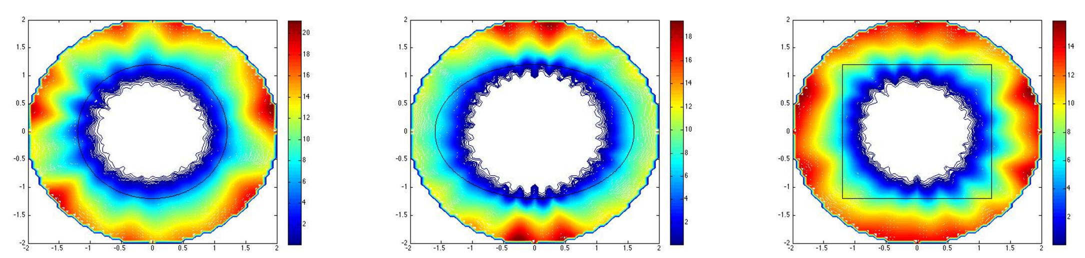
$$W(z) := \left[\sum_{j=1}^{M} \frac{|<\Phi_z, \phi_j>|^2}{|\lambda_j|}\right]^{-1}$$

for z varying in a region large enough to contain D. The cavity is the region where $W(z)$ is close to zero. In the following examples:

- D_1 is the disk of radius 2.
- C is the disk of radius 0.8.
- There are 30 incident point sources and 30 corresponding measurements equally distributed on ∂C.
- $A = [1.2 \quad 0; 0 \quad 1.5]$, $n = 0.8$, $k = 5$.



Reconstruction without noise.



Reconstruction with 0.1% noise