Bifurcations of a Prescribed Mean Curvature Equation

Nicholas D. Brubaker[1] , Alan E. Lindsay[2] , John A. Pelesko[1]

[1]Department of Mathematical Sciences, University of Delaware ({brubaker,pelesko}@math.udel.edu);
[2]Department of Mathematics, University of Arizona (alindsay@math.arizona.edu);

Introduction

Mathematical models of the form $Hu = f(u)$, where H is the mean curvature operator, are crucial in understanding capillary surfaces. They are particularly interesting mathematically due to the fact that many of their solutions sets undergo intriguing bifurcations (see e.g., [PX11]). Here, we study the solution set of the problem

$$\operatorname{div}\frac{\nabla u}{\sqrt{1+\varepsilon^2|\nabla u|^2}} = \frac{\lambda}{(1+u)^2}, \quad x\in\Omega; \quad u=0, \quad x\in\partial\Omega, \tag{1}$$

which derives from electrostatically deflecting a planar soap film. Specifically, we look at the two cases where $\Omega = [-1, 1]$ and $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$. Note that here λ, which characterizes the applied voltage, and ε, which characterizes the size the undeflected planar soap film, are nonnegative, dimensionless parameters.

One-dimensional

The 1D version of (1) reduces via symmetry to

$$\left(\frac{u'}{\sqrt{1+\varepsilon^2|u'|^2}}\right)' = \frac{\lambda}{(1+u)^2}, \quad 0<x<1; \tag{2}$$
$$u'(0) = u(1) = 0,$$

which has the first integral $\varepsilon^{-2}(1+\varepsilon^2|u'|^2)^{-1/2} - \lambda(1+u)^{-1} = E$. Therefore, in solving for u' and separating variables yields the following.

Lemma. *The values $(\lambda, \varepsilon, \alpha)$ give a solution u of the ordinary differential equation (2), with $u(0) = \alpha$, if and only if $T(\alpha; \lambda, \varepsilon) = 1$, where $E = \varepsilon^{-2} - \lambda/(1 + \alpha)$ and*

$$T(\alpha;\lambda,\varepsilon) := \int_\alpha^0 \frac{\varepsilon^3(\lambda + E(1+z))}{\sqrt{(1+z)^2 - \varepsilon^4(\lambda+E(1+z))}}\,\mathrm{d}z\,.$$

From this we have

Theorem 1. *There exists an $\varepsilon^* > 0$ such that*

(i) *if $\varepsilon \le \varepsilon^*$, then there exists a value $\lambda^*(\varepsilon)$ such that* (a) *for $\lambda \in (0, \lambda^*)$, (2) has exactly two solutions;* (b) *for $\lambda = \lambda^*$, (2) has exactly one solution;* (c) *for $\lambda > \lambda^*$, (2) has no solutions.*

(ii) *if $\varepsilon > \varepsilon^*$, then there exists three values λ_*, λ_{**} and λ^*, which depend on ε, such that* (a) *for $\lambda \in (0, \lambda_*] \cup [\lambda_{**}, \lambda^*)$, (2) has exactly two solutions;* (b) *for $\lambda \in (\lambda_*, \lambda_{**}) \cup \{\lambda^*\}$, (2) has exactly one solution;* (c) *for $\lambda > \lambda^*$, (2) has no solutions.*

Result: The solutions set of (2) undergoes a ***splitting bifurcation*** at $\varepsilon = \varepsilon^* \approx 2.857$, i.e., when ε transitions from less than or equal to to greater than ε^*, the upper solution branch splits into two parts (see the middle and bottom subfigures of Fig. 1).

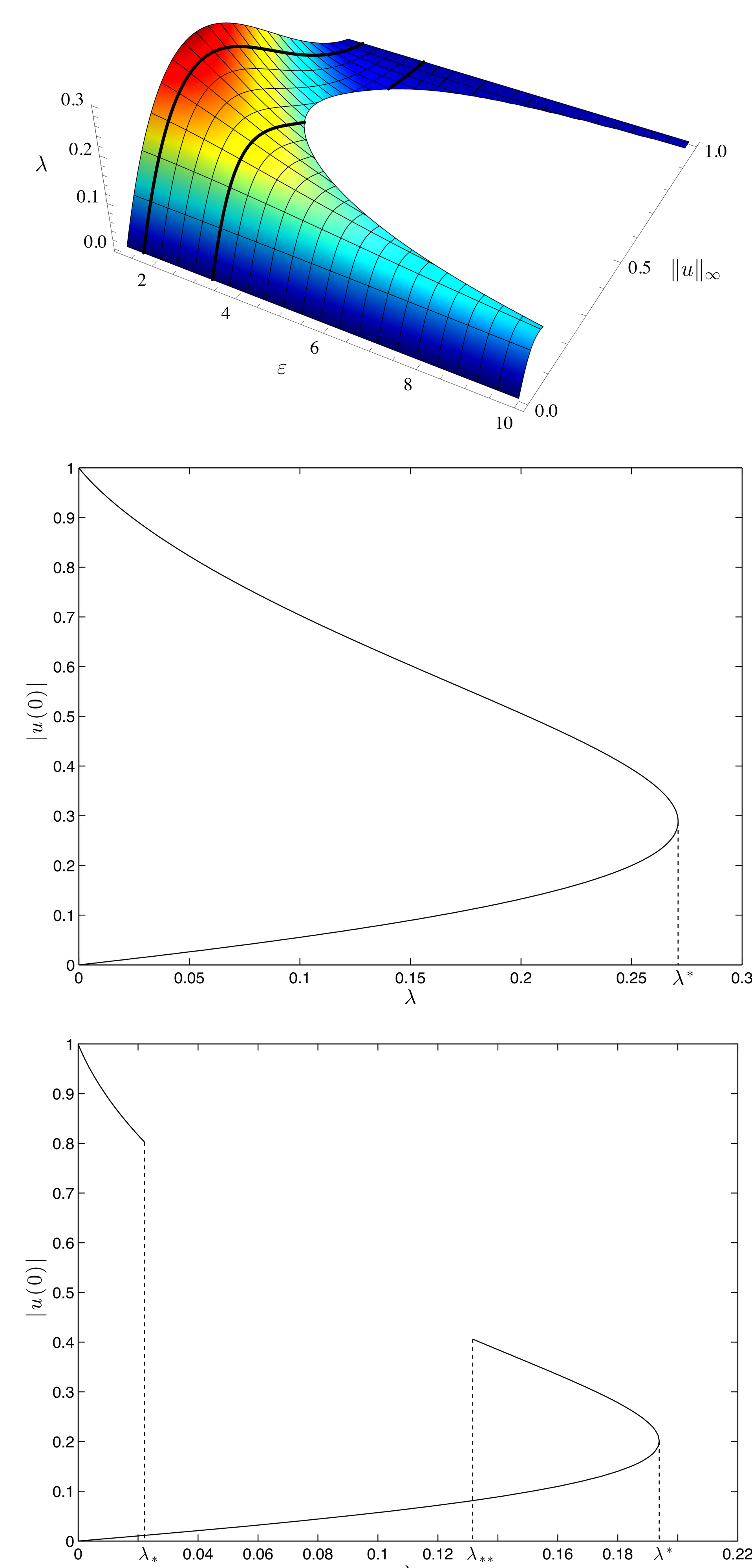

FIGURE 1. **Top:** Bifurcation surface, $\lambda(\varepsilon, |u(0)|)$, of (2) for $0 \le \varepsilon < 10$. The black contours represent solutions for $\varepsilon = 5/3$ (see **Middle**) and $\varepsilon = 10/3$ (see **Bottom**), which yield bifurcation curves that capture the qualitative shape described in the two cases of Theorem 1.

Two-dimensional

The 2D version of (1) with Ω equal to the unit disk reduces to

$$\frac{1}{r}\left(\frac{ru'}{\sqrt{1+\varepsilon^2|u'|^2}}\right)' = \frac{\lambda}{(1+u)^2}, \quad 0<r<1; \tag{3}$$
$$u'(0) = u(1) = 0.$$

This problem exhibits a ***dead-end bifurcation*** (see Figure 2).

- If $\varepsilon = 0$, then for all $\alpha \in (-1, 0]$ there exists a solution u of (3) such that $u(0) = \alpha$ [PB03].
- However, for $\varepsilon > 0$, there exists an $\alpha_*(\varepsilon) \in (-1, 0)$ such that if u is a solution of (3), then $|u(0)| < |\alpha_*|$.

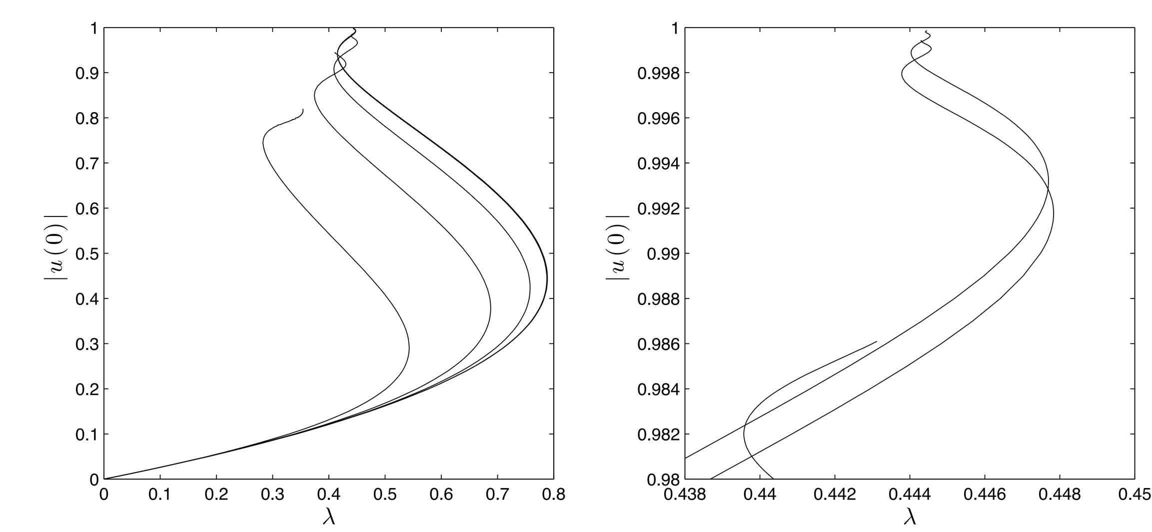

FIGURE 2. **Left:** Bifurcation curves of (3) computed for $\varepsilon = 0.05, 0.1, 0.5, 1, 2$ (from right to left). Note that at this scale the $\varepsilon = 0.05$ and $\varepsilon = 0.1$ curves appear equal. **Right:** Magnified portion of the left fig. Here, $\varepsilon = 0.05, 0.1, 0.5$ curves are seen. Note that all of the curves stop before $|u(0)|$ reaches 1.

Asymptotic analysis. To analyze the dead-end bifurcation for $\varepsilon \ll 1$, we look at (3) with the point constraint $u(0) = -1 + \delta$, for $0 < \delta \ll 1$. Since the problem involves two small parameters, the analysis must be performed in the distinguished limit $\varepsilon^2/\delta = \delta_0$ for $\delta_0 = \mathcal{O}(1)$; Expanding u and λ as $u \sim u_0 + \varepsilon^2 u_1$ and $\lambda \sim \lambda_0 + \varepsilon^2\lambda_1$ leads to a singular perturbation problem with a boundary layer of width $\mathcal{O}(\delta^{3/2})$ at $r = 0$. The leading order inner problem is

$$\frac{1}{\rho}\left(\frac{\rho w_0'}{\sqrt{1+\delta_0(w_0')^2}}\right)' = \frac{4}{9w_0^2}, \quad 0<\rho<\infty;$$
$$w_0(0) = 1, \quad w_0'(0) = 0,$$

whose far field behavior is

$$w_0 \sim \rho^{2/3} + \tilde{A}(\delta_0)\sin\left[\frac{2\sqrt{2}}{3}\log\rho + \tilde{\phi}(\delta_0)\right].$$

However there exists a value $\delta_0^* \approx 18.142468$ such that if $\delta_0 > \delta_0^*$, then no solution to this inner problem exists. Therefore asymptotic analysis is only valid for $\varepsilon^2/\delta \le \delta_0^*$. By performing matching we find that

$$\lambda \sim \frac{4}{9} - \delta\frac{4}{3}\tilde{A}\left(\frac{\varepsilon^2}{\delta}\right)\sin\left[\tilde{\phi}\left(\frac{\varepsilon^2}{\delta}\right) - \sqrt{2}\log\delta\right]$$

for $\varepsilon \ll 1$ and $\delta \ll 1$, with $\varepsilon^2/\delta \le \delta_0^*$, and since the asymptotic approximation fails beyond $\varepsilon^2/\delta = \delta_0^*$, we have the following result.

*For $\varepsilon \ll 1$, the **dead-end bifurcation point** of (3) has the asymptotic expansion*

$$|\alpha_*(\varepsilon)| \sim 1 - \frac{\varepsilon^2}{\delta_0^*},$$
$$\lambda_*(\varepsilon) \sim \frac{4}{9} - \varepsilon^2\frac{4\tilde{A}(\delta_0^*)}{3\delta_0^*}\sin\left[\tilde{\phi}(\delta_0^*) - \sqrt{2}\log\frac{\varepsilon^2}{\delta_0^*}\right],$$

where $\tilde{A}(\delta_0)$ and $\tilde{\phi}(\delta_0)$ are functions determined from the inner problem's far field behavior.

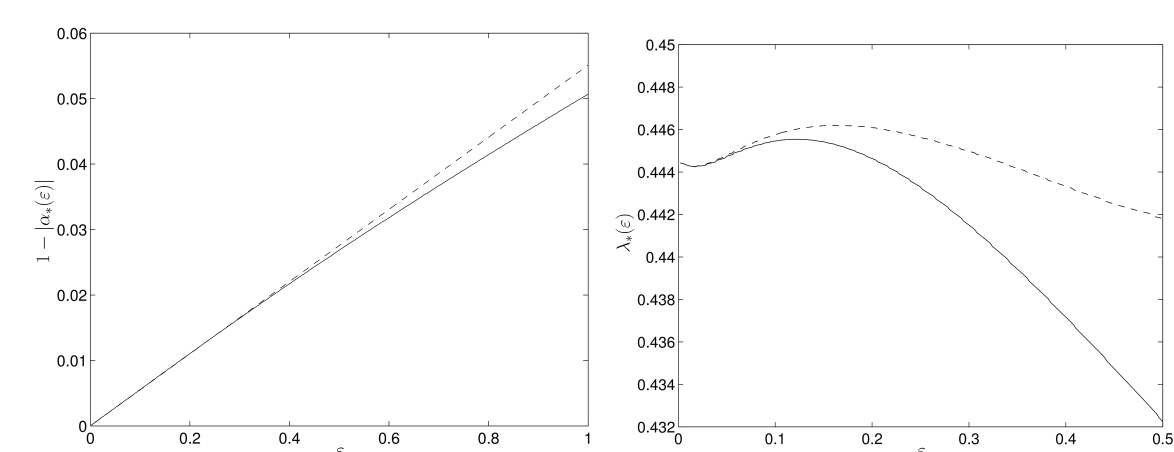

FIGURE 2. Comparison of the asymptotic prediction of the above result, (dashed line), of the dead-end point $(\lambda_*(\varepsilon), |\alpha_*(\varepsilon)|)$ with the full numerical computation (solid) for: **Left** the $\mathcal{O}(\varepsilon^2)$ correction of $|\alpha_*(\varepsilon)|$; **Right:** $\lambda_*(\varepsilon)$. Notice that the scale on the y-axis of the right figure is quite fine and so the agreement for $\lambda_*(\varepsilon)$ is in fact better than the figures makes it appear.

Acknowledgements

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