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Problem: **Frozen Shapes:**

Thin Nearly Flat Elastic Shells with Stretching and Bending

Introduction

Problems involving thin elastic plates/shells arise in the life sciences [1], aerospace [2], the paper industry [3], and with semiconductor wafers/ceramic sheets/glass sheets [4]. From an applied math perspective the problems are interesting for both their relevance and the distinct but not unfamiliar equations which arise. In this problem we are concerned with thin rectangular plates which are thin enough to be affected by thermal stresses due to a temperature distribution $\Delta T(x,y)$, but thick enough that the initial non-flat shape $\Delta z1(x,y)$, though small, provides a resistance to both bending and stretching.

We are also going to be interested in properly formulating and solving inverse problems, for example what thermal expansion distribution α T(x,y) might explain an observed non-flat shape z_meas(x,y). One direction this can lead is to the ambiguous problem of specifying a surface Z(x,y) given its Gaussian curvature G(x,y), so formulating the problem correctly is important.

We are going to be working on ***idealized model problems*** with simplifying assumptions. We will assume a thin rectangular plate with a uniform thickness, and look at ***scalings*** where the problem is interesting to the MPI group. The material is linear-elastic and the effect of thermal strain is idealized as a 'frozen' strain.

Other Background:

We'd like to apply spectral methods using Matlab [5] if feasible for at least model examples, and to understand the cases where such methods aren't appropriate.

References on thin plates, shell theory, and the role of Gaussian curvature are given in references [6], [7], [8]. A reference to thermal stresses in thin plates is [9]

Objective/Sub-Problems : connect the final shape z_tot(x,y) of a thin rectangular plate to the contributions of an initial non-flat shapeΔ z1(x,y) and the effect of thermal stress due to a temperature variation ΔT(x,y) . We are interested in the scalings where the deflections are 'large enough' that large-deflection corrections to the governing equation are needed. In this case there is coupling between two 4^{th} order non-linear PDEs.

The Sub-Problems below organize the questions but experience with MPI has shown that a lot of the benefit comes from a review of the formulation of the problem.

1. Solve for or estimate z_tot(x,y) with a known Δ z1(x,y) and known ΔT(x,y)
2. Given a known z_tot(x,y) and a known ΔT(x,y), can we solve for or estimate Δ z1(x,y) ?
3. Given a known z_tot(x,y) and a known Δ z1(x,y), can we solve for or estimate ΔT(x,y)?
4. If we only know z_tot(x,y), what additional information might be used to estimate both Δz1(x,y) and ΔT(x,y)?
5. A typical way of measuring z1(x,y) and T(x,y) will give a single function F(x,y) proportional to the total strain in the sheet when pressed flat – it is the sum of two terms, one proportional to the Gaussian curvature of z1 and a second proportional to the Laplacian of T(x,y). The most realistic problem in this model problem set is to take the total shape z_tot(x,y) and the measured function F(x,y) and ask how one might estimate the components z1(x,y), T(x,y). (This is similar to the classical Dirichlet problem of figuring out the shape given only the Gaussian curvature)
6. Sag Problem : A related sub-problem is to calculate the sag of a thin plate with frozen shape z1(x,y) when resting on 3 or more pins, using the Matlab spectral methods suggested in [5] if feasible or another Matlab implementation. Most examples of sagging plates assume that all points on the boundary are either fixed or simply supported, which dramatically simplifies the problem. Is there a good way to handle the problem with the edges free. In this case we can first look at (6a) the small deflection problem (sheet is stiffer because it is thicker or smaller in size) and then worry about 'large deflection effects' as (6b)

Governing Equations for a 'frozen but flexible' sheet

The governing equation for a thin elastic plate with no embedded thermal strains, deformed under pressure (like gravity) is a standard linear PDE involving a biharmonic equation [6 p.42]

$$(1) \quad D\nabla^4 w(x,y) = p(x,y)$$

This basic equation is really only valid when the deformation is well less than the thickness of the plate. We are interested in including the effect of a 'frozen in' thermal strain as well as the effect of modest deformations requiring the 'stretching' of the sheet. We are also interested in the initial shape being non-flat – technically this makes it a 'shell' and not a 'plate' in the literature.

The equation for the 'large deflection' of a thin plate includes a second equation for the stress which develops from stretching the sheet, and the effect of this stress on the deflection[6, p.52-53]. These are 2nd order if the deflection is small enough. The term "g(x,y)" is the Gaussian curvature of the deformed sheet:

$$(2a) \quad \nabla^4 \phi = E\{g(x,y)\} = E\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}$$

$$(2b) \quad D\nabla^4 w = p + h\left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right)$$

The equations for a thin elastic plate which has some thermal strain $\alpha\Delta T(x,y)$ requires the same additional equation for the in-plane stresses, now including the effect of thermal strain. The stresses satisfy a compatibility condition so that they can be derived from a scalar field $\phi(x,y)$, the Airy stress function. If the deflection were small the equations would be:

$$(3a) \quad \nabla^4 \phi = E\nabla^2(\alpha\Delta T)$$

$$(3b) \quad D\nabla^4 w = p$$

The problem we are interested in has

a. the thermal strain from (3a)

b. The 'large deflection' effect from (2a). Both effects are important to ϕ and equation (2b).

c. We want to assume the initial shape is non-flat .

Definitions

1. h = Thickness
2. D = Eh^3 / (1-v^2) = plate stiffness. E = Young's Module, v = Poisson's Ratio
3. α = coefficient of thermal expansion
4. ϕ = Airy Stress function
5. P = pressure, T = temperature, w = deflection

GENERAL REFERENCES

[1] –[4] General References

[1a]Sharon, Marder, and Swinney, "Leaves, Flowers, and Garbage Bags: Making Waves" ***American Scientist*** May-June 2004 pp.254-261.

[1b] Sharon, Roman, Marder, Shin, Swinney, "Buckling cascades in thin sheets", ***Nature*** vol.419 (10Oct2002) p.579.

[2] for example, Buttzao, Frediani, editors, ***Variational Analysis and Aerospace Engineering: Mathematical Challenges for Aerospace Design***. New York: Springer, 2012.

[3] for example, Niskanen, editor, ***Mechanics of Paper Products***. New York: De Gruyter, 2011.

[4] for example (film on semiconductor wafer), Huang and Rosakis, "Extension of Stoney's formula to non-uniform temperature distributions in thin film/substrate systems. The case of radial symmetry", ***J. Mech. Phys.Solids***, vol.53(2005)pp.2483-2500.

[5] Spectral Methods: L. N. Trefethen, ***Spectral Methods in MATLAB***. Philadelphia: SIAM, 2000.

[6] –[9] references on thin plates and shell theory, thermal stresses

[6] Landau and Lifshiftz, ***Theory of Elasticity 3rd Edition***. New York: Elsevier, 1986.

[7] Calladine, ***Theory of Shell Structures***. New York: Cambridge University Press, 1983.

[8] Ventsel and Krauthammer, ***Thin Plates and Shells: Theory, Analysis, and Applications***. New York, Marcel Dekker, 2001

[9] Boley and Weiner, ***Theory of Thermal Stresses***. New York: Dover reprint of 1985 edition.