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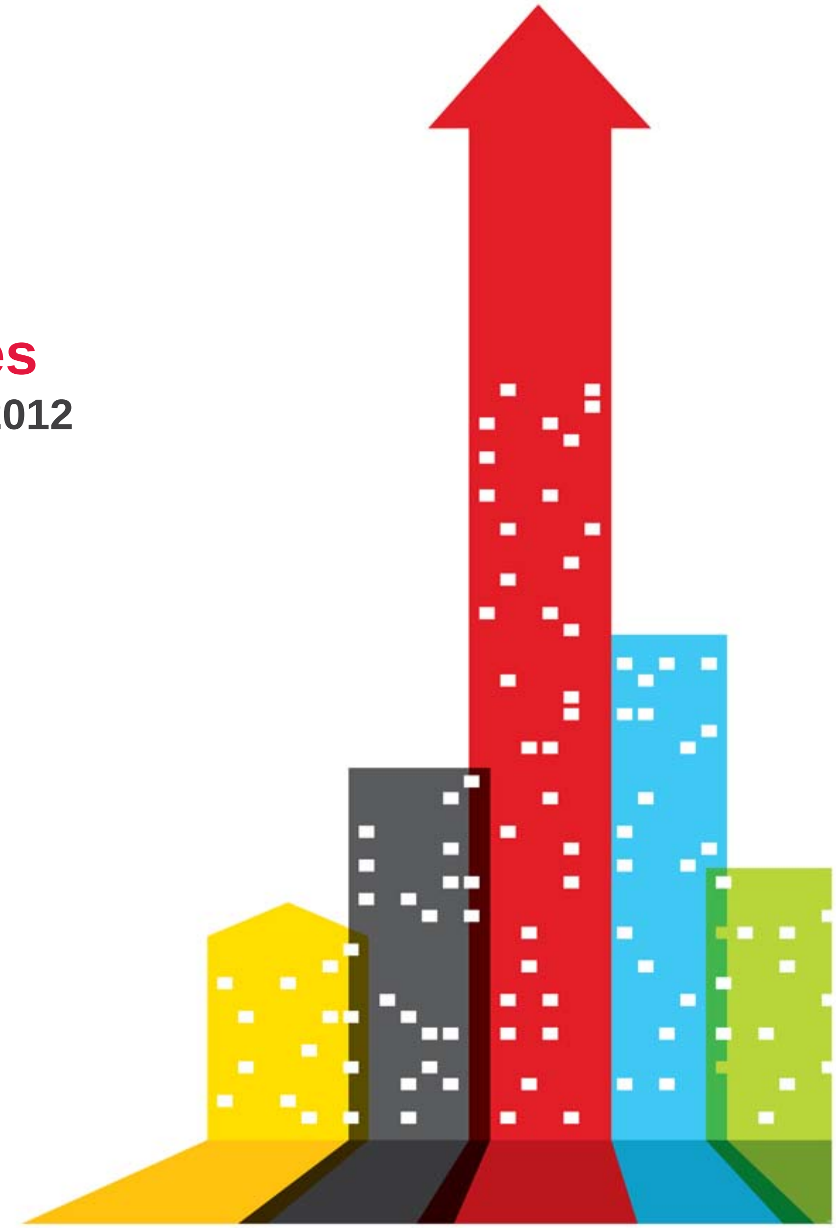
Approximating Correlation Matrices

Mathematical Problems in Industry Workshop 2012



William Morokoff
Managing Director
Quantitative Analytics Research Group

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McGRAW-HILL

Roadmap for Correlation Discussion

Themes and Summary of Key Points	
Standard & Poor's Overview	• Company profile • Quantitative analytics at S&P
Portfolio Modeling and Correlation	• Modeling dependence with correlation
Representing Correlation: Factor Models	• Properties of correlation matrices • Empirical correlation estimates • Block correlation representation • Factor models • Optimal factor model approximations
Localized Factor Models	• Localized Factor Models • Best approximation of localized one factor models • Semi-analytic methods for localized one factor models
Open Research Questions	• Optimal approximations • Computational methods

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S&P Overview

S&P Overview

- Standard and Poor's is a leading provider of independent credit analysis, and a key source for financial market intelligence.
- Best known for S&P 500 Index and Credit Ratings
- Wholly owned subsidiary of McGraw-Hill
- About 10,000 employees worldwide with around $2.9 Billion in combined revenue for Standard & Poor's Ratings Services and S&P Capital IQ.
- Brands include Compustat, CUSIP, Capital IQ, ClariFI, Risk Solutions, GICS, RatingsDirect, IMAKE, R2

Role of Quantitative Analytics Research Group at S&P

- **Mandate: Support S&P Ratings Services and S&P Capital IQ with quantitative expertise to grow the businesses and develop products and services that enhance S&P's position as a respected voice in the capital markets.**
- **Priorities:**
  - Quantitative Support for Ratings Methodology
  - Model Development
  - Quality and Efficiency
  - Quantitative Strategic Vision across Businesses
  - Thought Leadership
  - Business Advisory and Training

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Portfolio Modeling and Correlation

Credit Portfolio Modeling

$$\Pi(t) = \sum_{i=1}^{N} \omega_i \; V_i(t)$$

$$R_{\Pi}(T_0, T_H) = \sum_{i=1}^{N} \hat{\omega}_i \; R_i(T_0, T_H)$$

$$EL = E\left(R_{ref} - R_{\Pi}\right)$$

$$UL = \sigma\left(R_{ref} - R_{\Pi}\right)$$

$$P\left(R_{\Pi} < R^{*}(\alpha)\right) = \alpha$$

Credit Portfolio Modeling

- Distribution of returns for individual exposures is a key component of determining the portfolio return distribution.
- However, the joint dependence of returns is crucial for determining the portfolio potential for large losses.
- The joint behavior is often modeled by normalizing the distribution of individual returns and estimating a multi-variate distribution for the normalized returns – often referred to as latent variables or asset returns.

$$z_i = \Phi^{-1}\left(F_i\left(R_i\right)\right)$$

- Specifying a correlation matrix (symmetric, pos def with ones on diagonal)

$$P = E\left(zz^T\right)$$

or more generally a copula function determines the dependence of the asset returns.

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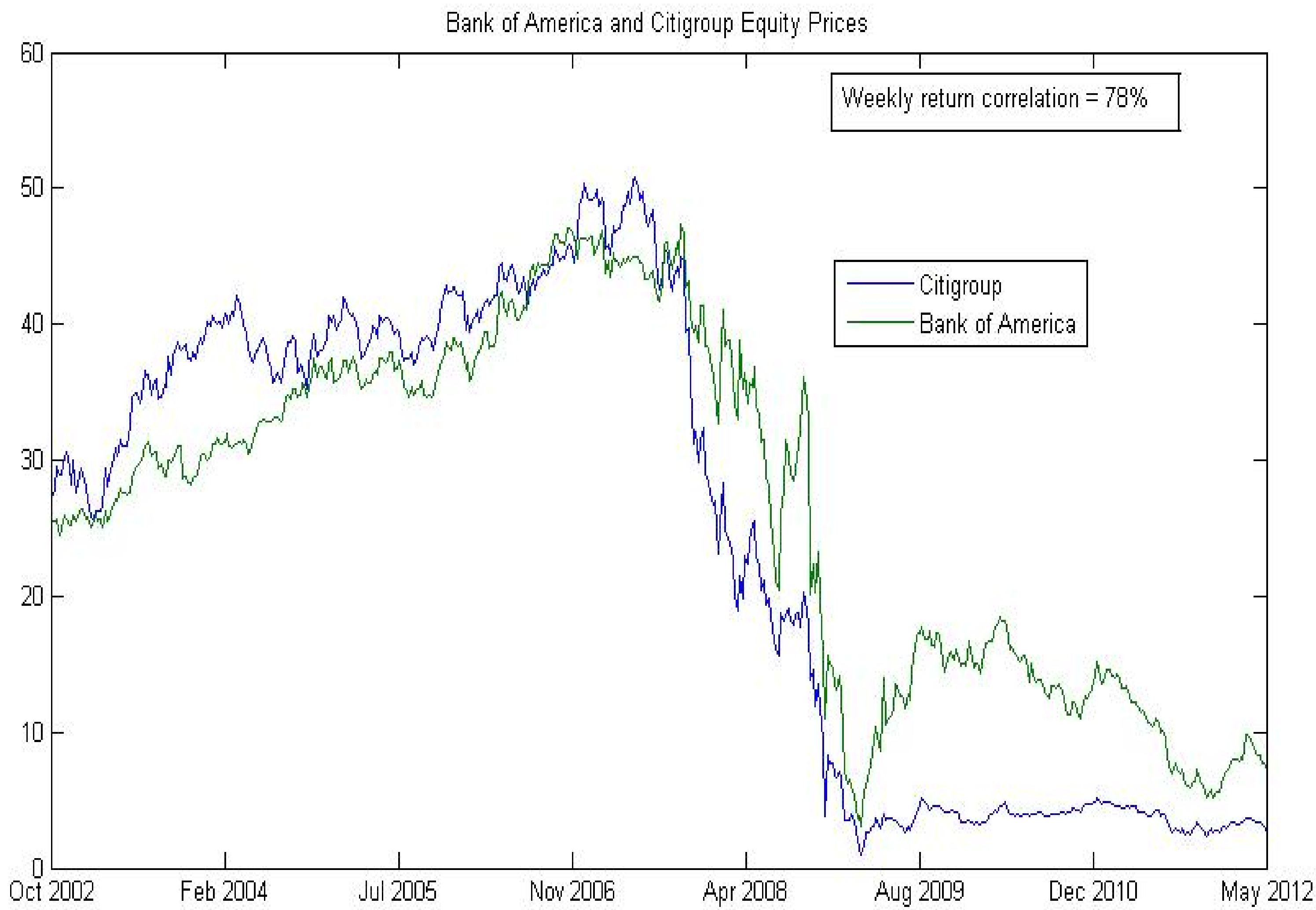
Equity Return Correlation Example



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Representing Correlation: Factor Models

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Correlation Matrix Properties

- Symmetric
- Positive Definite
- Ones on the diagonal
- Off-diagonal elements represent pair-wise correlation of two assets in the portfolio
- Correlations are usually positive – particularly for firm asset returns in credit modeling
- Example: This doesn't work:

$$\begin{bmatrix} 1 & .1 & .9 \\ .1 & 1 & .9 \\ .9 & .9 & 1 \end{bmatrix}$$

Empirical Correlation Estimates

- **If time series of ‘asset returns’ is available or can be proxied by equity returns or other data, pair-wise correlations can be estimated.**
- **Estimate of full correlation matrix may be somewhat unstable due to missing data and large number of correlation parameters being estimated – N*(N-1)/2 for N firms.**
- **For large portfolios, describing dependence through a full correlation matrix is impractical.**

Factor Models

- **Gaussian Copula Factor Model – Standard Model**
  - Normalized ‘Asset Return’ to Horizon is modeled as a standard Normal random variable.
  - Asset return is decomposed into systematic risk component and idiosyncratic risk component.
  - Percentage of variance related to systematic risk is ‘R-squared’.
  - Systematic risk is described by one or more independent standard normal variates, common to all exposures. Each exposure is assigned a set of weights on the factors. The idiosyncratic risk is modeled as standard normal, independent of all factors and other exposures.

$$z_i = \sqrt{\rho_i}\ \beta_i^T \varepsilon_F + \sqrt{1-\rho_i}\ \varepsilon_{I,i} \qquad \beta_i^T \beta_i = 1$$

$$\rho_{ij} = \sqrt{\rho_i \rho_j}\ \beta_i^T \beta_j$$

Factor Models

- In matrix notation, the factor model is

$$z = \Gamma^{1/2} B \epsilon_F + [I - \Gamma]^{1/2} \epsilon_I$$

- The correlation matrix can be represented as

$$P = \Gamma^{1/2} B B^T \Gamma^{1/2} + I - \Gamma$$

Block Correlation Matrix

$$P = \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{1n}^T & \cdots & P_{nn} \end{bmatrix} \qquad \mathrm{P}_R = \begin{bmatrix} \rho_{11} & \cdots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{1n} & \cdots & \rho_{nn} \end{bmatrix}$$

$$P_{ii} = (1-\rho_{ii})\,\mathrm{I}_{N_i} + \rho_{ii}\mathrm{E}_{N_i} \qquad N_i \times N_i$$

$$P_{ij} = \rho_{ij} E_{N_i \times N_j} \qquad N_i \times N_j$$

The reduced correlation matrix P_R is not necessarily positive definite

$$P_R^* = P_R + diag\left(\left[\frac{1-\rho_{11}}{N_1}, \cdots, \frac{1-\rho_{nn}}{N_n}\right]\right) > 0$$

Block Correlation Matrix – Factor Model Representation

- **If exposures are classified into groups based on sector, geography, size, etc., and correlations between all members of two groups are assumed the same, the resulting correlation matrix has a block structure.**
- **The n diagonal blocks are correlation matrices with a constant correlation in each block. Each block has N_i exposures, i = 1 … n.**
- **The off-diagonal blocks have identical elements within a block.**
- **In order to reduce the computation burden of working with large matrices, block correlation matrices can be factored as:**

$$z_{ij} = \sqrt{\rho_i + (1-\rho_i)/N_i}\ \beta_i^T \varepsilon_F + \sqrt{1-\rho_i}\ \left(\varepsilon_{ij} - \bar{\varepsilon}_i\right) \qquad i = 1 \dots n, \quad j = 1 \dots N_i$$

$$\beta_i^T \beta_i = 1 \qquad \beta_i - n \times 1 \text{ vector}$$

$$\varepsilon_F - n \times 1 \text{ vector of independent standard Normal variates}$$

$$\varepsilon_{ij} - \text{ iid standard Normal variates}$$

$$\bar{\varepsilon}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} \varepsilon_{ij}$$

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Optimal Factor Model Approximation

- **Andersen, Sidenius and Basu (Risk, Nov 2003) show how to find the best K factor approximation to a given correlation matrix P in the sense of minimizing the matrix difference in the Frobenius norm through principal component analysis (PCA).**
- **Given an estimate of the R-squared diagonal matrix, carry out the following iteration:**

$$Q_i D_i Q_i^T = P - I + \Gamma_i$$

$$\Gamma_{i+1} = diag\left(Q_i D_i^{(K)} Q_i^T\right)$$

$$\Gamma_i \to \Gamma_{opt}^{(K)}$$

$$B_{opt}^{(K)} = \left(\Gamma_{opt}^{(K)}\right)^{-1/2} Q_{opt}^{(K)} \sqrt{D_{opt}^{(K)}}$$

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Localized Factor Models

Localized One Factor Model

- Assume that there are one global factor and K sector factors.
- A 'localized one factor model' is a factor model where each exposure weights on the global factor and exactly one sector factor.

$$z_i = \sqrt{\rho_i}\left(\beta_i^0 \varepsilon_0 + \beta_i \varepsilon_{k(i)}\right) + \sqrt{1-\rho_i}\,\phi_i$$

$$\left(\beta_i^0\right)^2 + \left(\beta_i\right)^2 = 1$$

- If all exposures in one sector have the same R-squared value and same factor loadings, then this corresponds to a block correlation matrix. However, not every block correlation matrix can be expressed as a localized one factor model.

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Localized Two Factor Model

- **Assume that there are one global factor and K sector factors and J country factors for a total of (K+J+1) factors.**
- **A ‘localized two factor model’ is a factor model where each exposure weights on the global factor, exactly one sector factor and exactly one country factor.**

$$z_i = \sqrt{\rho_i}\left(\beta_i^0 \varepsilon_0 + \beta_i^S \varepsilon_{k(i)}^S + \beta_i^C \varepsilon_{j(i)}^C\right) + \sqrt{1-\rho_i}\,\phi_i$$

$$\left(\beta_i^0\right)^2 + \left(\beta_i^S\right)^2 + \left(\beta_i^C\right)^2 = 1$$

- **If all exposures in one sector have the same R-squared value and same factor loadings, then this corresponds to a block correlation matrix. However, not every block correlation matrix can be expressed as a localized two factor model.**

Optimal Localized One Factor Approximation

- **For a block correlation matrix P with K diagonal blocks and homogeneous correlations within each block, let ρ_k be the diagonal block correlations and ρ_{jk} be the off-diagonal block correlations.**
- **For each exposure in sector k (k = 1 … K), set the R-squared value to ρ_k**
- **Construct the correlation matrix corresponding to the ρ_{jk} . Find the best single factor approximation using PCA technique. Set**

$$\beta_k^0 = \sqrt{\frac{\Gamma_{opt,k}^{(1)}}{\rho_k}} \qquad \beta_k = \sqrt{1-\left(\beta_k^0\right)^2}$$

- **If reduced correlation matrix is not positive definite, need to modify.**

Semi-Analytic Methods for Localized One Factor Model

- For a portfolio with exposures in K sectors, the total loss is

$$L = L_1 + \ldots + L_K$$

- Conditional on the global factor, these sector losses are independent, and each sector is described by a single factor model, so the usual semi-analytic methods can be applied on a sector-by-sector basis:

$$P(L) = E_{\varepsilon_0}\left[\sum_{L_1+\cdots L_K = L} P\left(L_1, \cdots, L_K \mid \varepsilon_0\right)\right]$$

$$P(L_1, \cdots, L_K \mid \varepsilon_0) = \prod_{k=1}^{K} P\left(L_k \mid \varepsilon_0\right)$$

$$P\left(L_k \mid \varepsilon_0\right) = E_{\varepsilon_k}\left[P\left(L_k \mid \varepsilon_0, \varepsilon_k\right)\right]$$

- Alternative: Saddle Point Approximation

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Open Research Questions

Optimal Approximations

- For a block correlation matrix, how does the k factor PCA optimal approximation compare with the block correlation factor model representation reduced to k factors?
- For a block correlation matrix, how does the k factor PCA optimal approximation compare with the optimal localized one-factor approximation?
- Can a better localized one-factor model approximation be found that incorporates the number of assets per group?
- How can the optimal localized two factor model be determined?
- Can optimal be measured in terms of portfolio risk metrics as opposed to measured in a matrix norm?

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Computational Methods

- **For a localized one factor approximation, what are the challenges and limitations of implementing a semi-analytic numerical approach? How efficient is this approach relative to a full Monte Carlo simulation?**
- **To what extent is it possible to adapt saddle point approximation methods to localized one factor models?**

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