# The Dynamics of a Model of A Computer Protocol

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The Internet has become a central means of communication on a global scale. Its emergence over the last two decades has been facilitated by the introduction of congestion control methodologies such as TCP, that allow millions of users to share network resources without causing congestion collapse. Beginning with the work of Frank Kelly [2], researchers have discovered that many currently used protocols designed to minimize congestion and to provide some notion of "fairness" can be viewed as a distributed, iterative algorithm for solving a global optimization problem. With this it is now possible to use mathematical analysis to help study, characterize and design protocols.
The model we will study is based on the work of Wang, Li, Low and Doyle [5] who developed distributed algorithms for maximizing network utility through joint control of congestion and routing. They considered two routing allocation strategies. The first, multipath routing occurs when a number of paths connecting a source and destination may be available and the goal is to assign traffic and bandwidth rates along the routes so that network utility is maximized. The second approach, single path routing, is on the other end of the spectrum of path specification. Again there are multiple paths linking a source and destination, but only a single path is selected, for example a path with minimum cost. Wang et al proved that there is no duality gap for the multipath optimization problem. However in general there is a duality gap for the single path problem. They established the equivalence between the existence of an equilibrium for the algorithm and the absence of a duality gap for the single path problem. Yet even though there is an equilibrium it can be unstable. Wang et al proposed the addition of a static component to the link costs in order to stabilize it. The following equations model a congestion control protocol with a random route allocation scheme that relaxes the requirement for a single path of minimum cost, but also reduces the number of eligible paths that would have to be considered in the multipath case. This is done by constraining the entropy of the route distribution. By varying the entropy, the route allocation scheme dials between single path and multipath routing. We want to determine the dynamics of this model as the entropy varies.
We represent a computer network as a planar graph with nodes representing locations in the network. In the literature, the total amount of bandwidth to send information from one part of the network to another is controlled by an agent called a source. There are several routes a source can take to its destination and we assume each route is defined by a set of uni-directional links indexed by $l = 1, \cdots L$. These links are represented in the graph as weighted edges, where an edge weight is the link capacity. Sources are indexed by $s$, and traffic is assigned to route $r \in R(s)$ with probability $\beta_{rs}$ where $R(s)$ is the set of all routes used by source $s$. If $p_l(k)$ is the link cost at time $k$ and $c_l$ is the capacity of the $l$th link then the equations of the model are

$$p_l(k+1) = \mathcal{H}\left(p_l(k) - h\left[c_l - \sum_s x_s(k) \sum_{r \in R_s(l)} \beta_{sr}(k)\right]\right) \quad l = 1, \cdots L \tag{1}$$

$$\beta_{sr}(k) = \exp(-\gamma_s(k) d_r(k))/Z_s(k) \tag{2}$$

where $\mathcal{H}$ is the Heaviside function, $d_r(k) = \sum_{l \in r} p_l(k)$ is the cost of route $r$ at time $k$, $h$ is a step size and

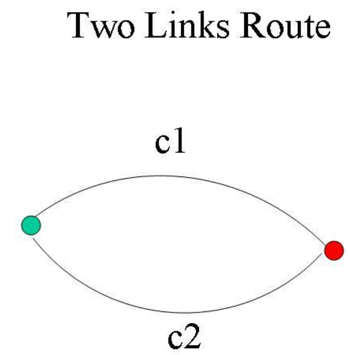


Figure 1: A simple example: the two link network.

$Z_s(k) = \sum_{r \in R(s)} exp(-\gamma_s(k) d_r(k))$. The variable $\gamma_s(k)$ is the solution of the implicit equation,

$$\gamma_s(k) D_s(k) + \log(Z_s(k)) = h_s, \quad D_s = \sum_{r \in R(s)} \beta_{sr}(k) d_r(k) \tag{3}$$

The model equations are completed by a relation between the bandwidth rate $x_s(k)$ and $D_s(k)$, the mean route cost at time $k$ for positive constants $w$ and $M$.

$$x_s(k) = \min\left( \left( \frac{w}{D_s(k)} \right)^{1/2}, \; M \right) \tag{4}$$

Equations (2) and (3) force the route distribution $\beta_{sr}(k)$ to be the unique distribution of entropy $h_s$ with the smallest <u>mean</u> route cost at each time step $k$. Recall that for any probability distribution $\beta_{\mathbf{s}} = \{\beta_{sr}\}_{r \in R(s)}$ the entropy of the distribution is

$$\mathsf{H}(\beta_{\mathbf{s}}) = -\left( \sum_{r \in R(s)} \beta_{sr} \log \beta_{sr} \right). \tag{5}$$

Thus the condition on the route distributions is $\mathsf{H}(\beta_{\mathbf{s}}) = h_s$. This constant entropy requirement will constrain the set of values $\{p_l \mid l = 1 \cdots L\}$ for which a bounded $\gamma_s(k)$ exists. The precise set depends on the network topology and capacity of the links. To illustrate these points we end with a practice problem for a two link topology depicted as a graph with two nodes, an origin and destination node and two links joining them. (see figure 1). Each route consists of a single link labeled 1 or 2. The equations (1)-(4) for this system are:

$$\begin{aligned} p_1(k+1) &= \mathcal{H}\left(p_1(k) - h\left[c_1 - x_s(k)\beta_{s1}(k)\right]\right) \\ p_2(k+1) &= \mathcal{H}\left(p_2(k) - h\left[c_2 - x_s(k)\beta_{s2}(k)\right]\right) \\ \beta_{s1}(k+1) &= \exp\left(-\gamma_k p_1(k)\right)/Z_s(k) \\ \beta_{s2}(k+1) &= \exp\left(-\gamma_k p_2(k)\right)/Z_s(k) \end{aligned} \tag{6}$$

where $c_1, c_2$ are the link capacities and $\beta_{s1}$, $\beta_{s2}$ are the probabilities that traffic is directed to links 1 or 2 respectively. If we note that in this case, the route costs at time $k$ are $p_1(k)$ and $p_2(k)$ respectively and the mean route cost is $D_s(k) = \beta_{s1} p_1(k) + \beta_{s2} p_2(k)$, then $\gamma_k$ is the solution of the implicit equation (3) and $x_s(k)$ is given by equation (4).

In [1] we analyzed (6) for a range of parameters that made (6) more tractable. We first observed that if $h_s < \log 2$ then there is no bounded solution of equation (3), for any values in the set $\{p = (p_1, p_2) \mid p_1 = p_2\}$. Thus we chose initial values of $p$ in the region $\{p = p_1, p_2) \mid p_1 < p_2 \}$. Fix the link capacities so that $c_1/c_2 = \lambda > 1$. Our choice of capacities automatically provide a reference value of $h_s$ around which we can perturb (see [1] ). In light of this, it turns out that changing the route allocation distribution is a way to change $h_s$ Given an $h_s$ there is a unique $\beta$ (up to interchange between $\beta$ and $1-\beta$), such that $h_s = \mathsf{H}(\beta, 1-\beta)$. Let

$$\beta_1^{(h_s)}/\beta_2^{(h_s)} = \lambda + \mu. \tag{7}$$

If $\mu = 0$ then $h_s$ is the entropy of the route distribution defined by the link capacities $c_1, c_2$. If $\mu < 0$ then one can show that $h_s$ increases and the allocation approaches the equal weight multipath pattern. For $\mu > 0$, the entropy decreases and we approach the single path scheme, i.e. the selection of the route with the largest capacity and lowest mean price and therefore the largest bandwidth rate.
Some questions we would like to answer are

- Can we rigorously establish that the equilibrium points of (6) are solutions of the original optimization problem when one of the coordinates vanishes? Does the method of proof generalize to other topologies?
- Is there a stable (sub-gradient?) iterative method for obtaining optimal solutions of the problem when $\mu < 0$ if they exist?

# References


[1] F. Hunt, V. Marbukh, "A Model of Joint Congestion Control and Routing Through Random Assignment of Paths" in preparation 2009.

[2] F.P. Kelly, A.K.Maulloo, D.K.H. Tan "Rate control in communication networks: shadow prices, proportaion fairness and stability." Journal of the Operational Research Society, Vol. 49, pp. 237-252, 1998.

[3] S.H. Low, D. E. Lapsley "Optimization Flow Control I: Basic Algorithm and Convergence" IEEE/ACM Transactions on Networking Vol. 7, No. 6, pp.861-874, 1999
available at http://netlab.caltech.edu/pub.

[4] V. Marbukh, S. Klink "Decentralized Control of Large Scale Networks as a Game with Local Interactions:Cross Layer TCP/IP Optimization" ValueTools07.

[5] J.Wang, L. Li, S.H. Low, J.C.Doyle "Cross-Layer Optimization in TCP/IP networks" IEEE Transactions on Networking, Vol. 13, No. 3, pp. 568-582, 2005.