Ovoids in the Triality Quadric

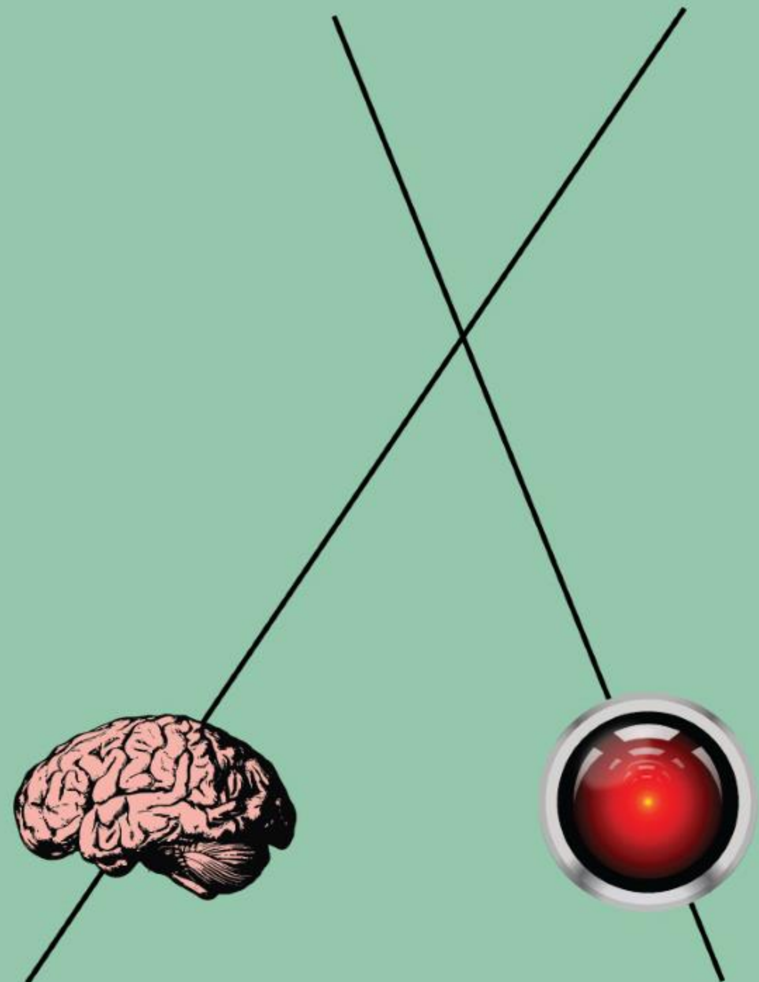
G. Eric Moorhouse

Department of Mathematics
University of Wyoming

FGEC 2019

Axiom 1

Axiom 2



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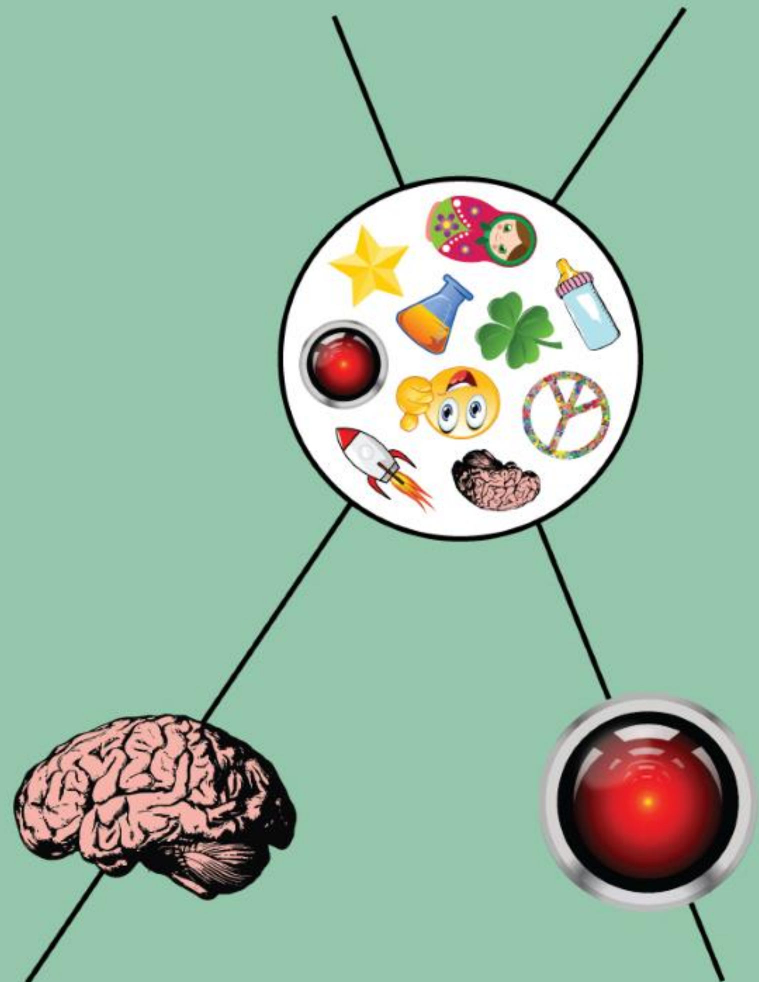

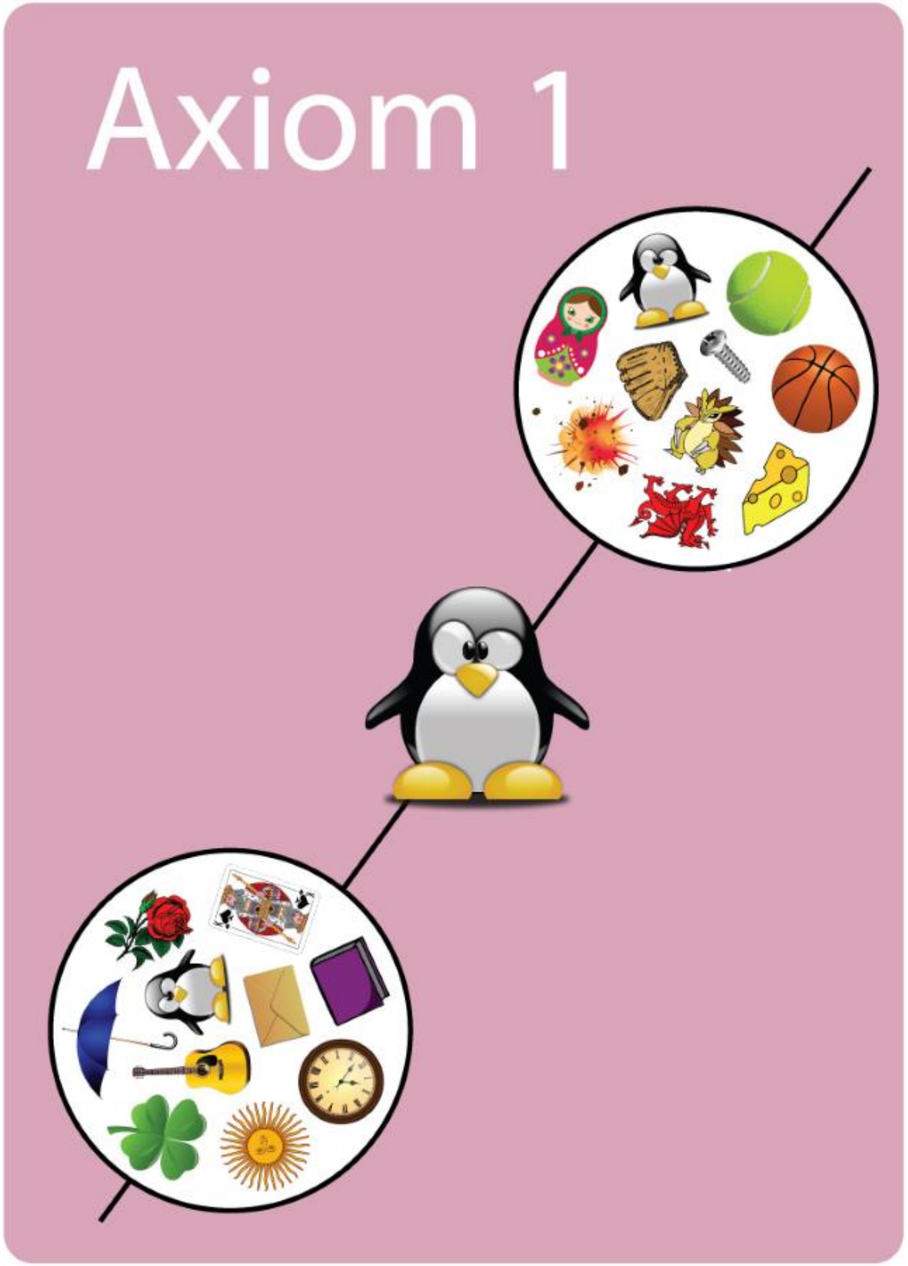
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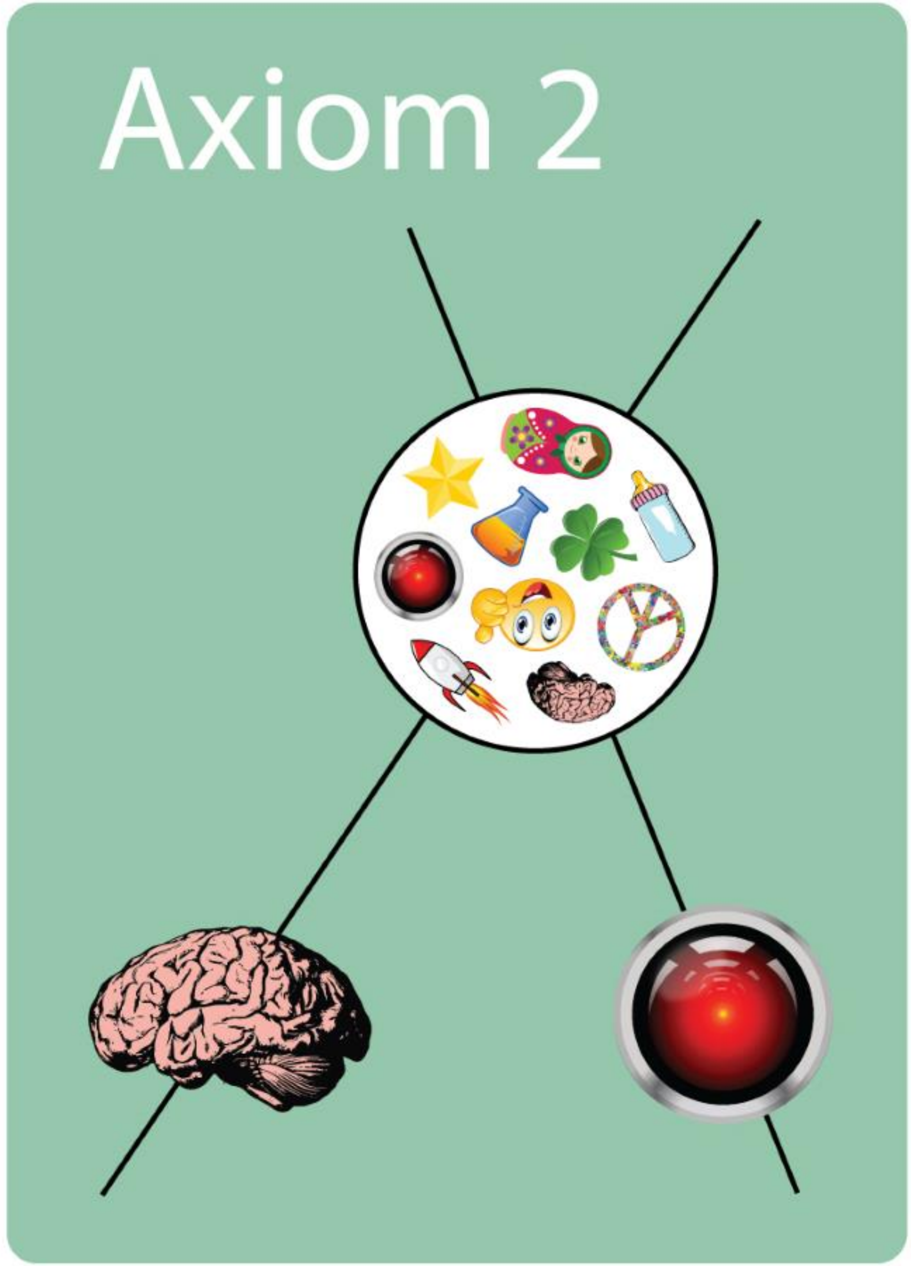
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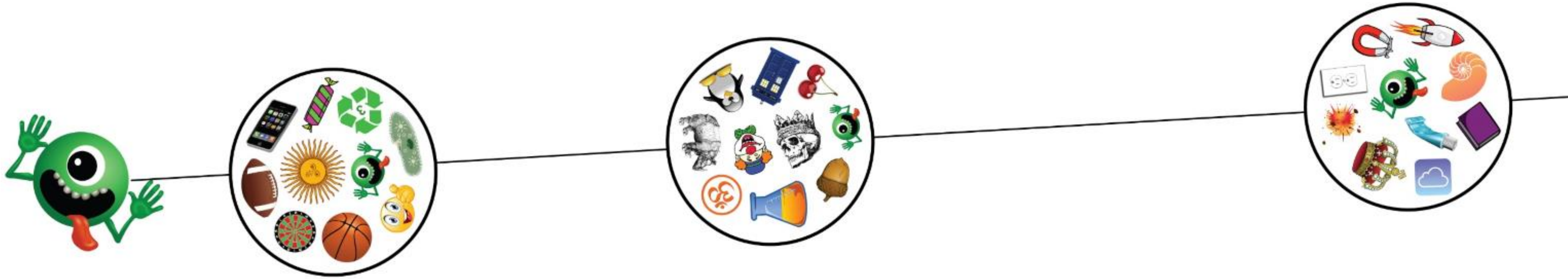
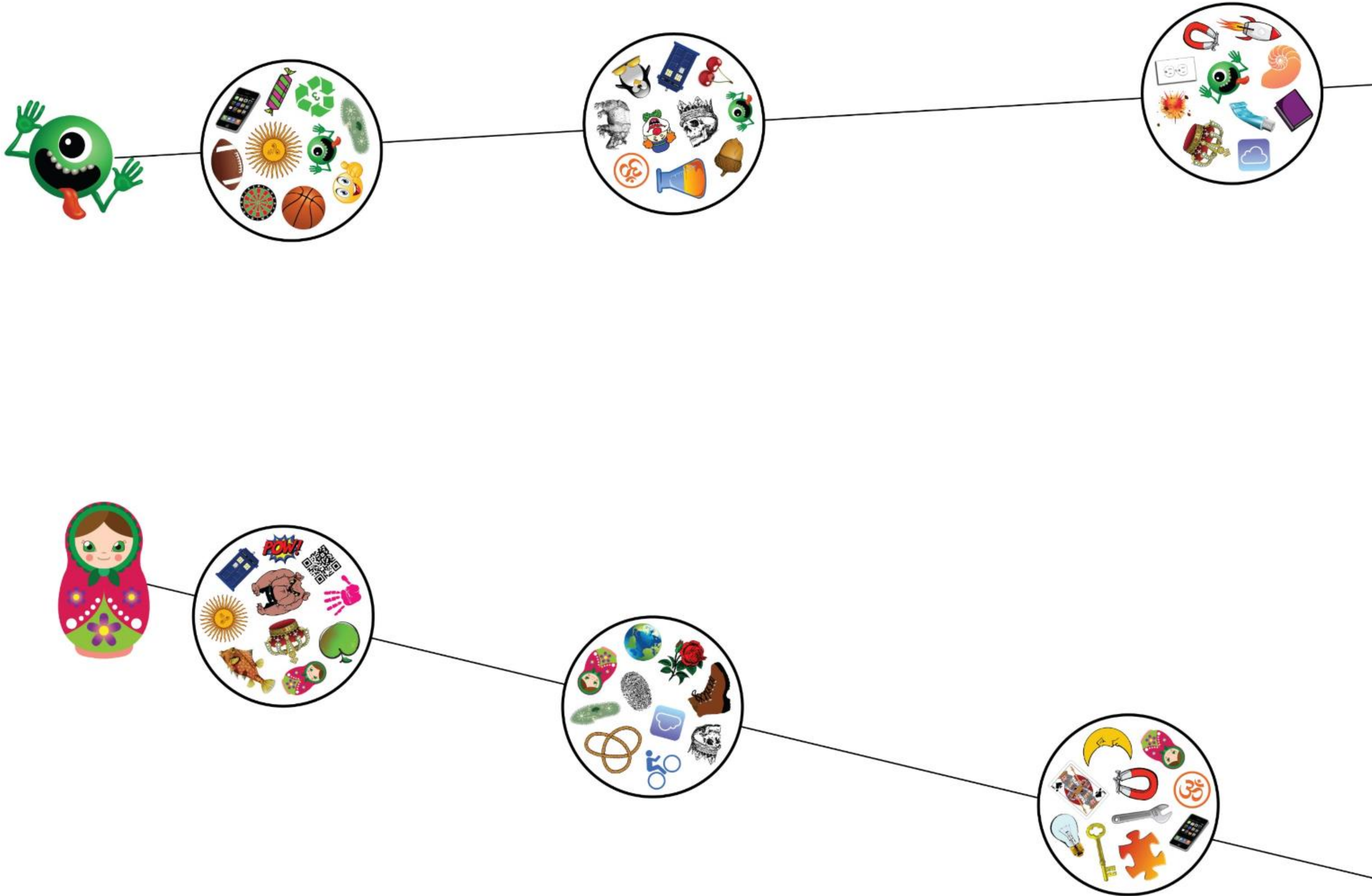
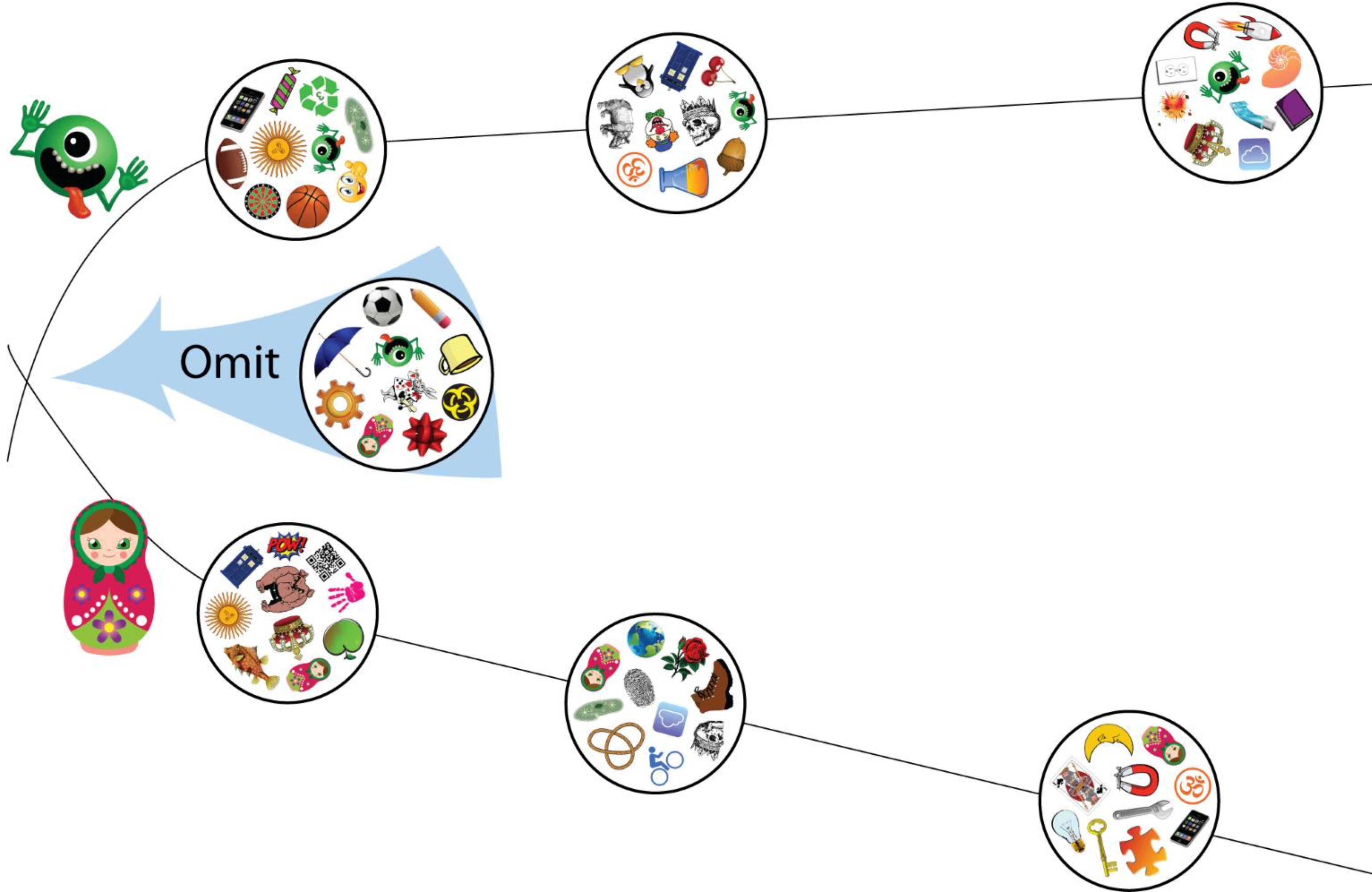
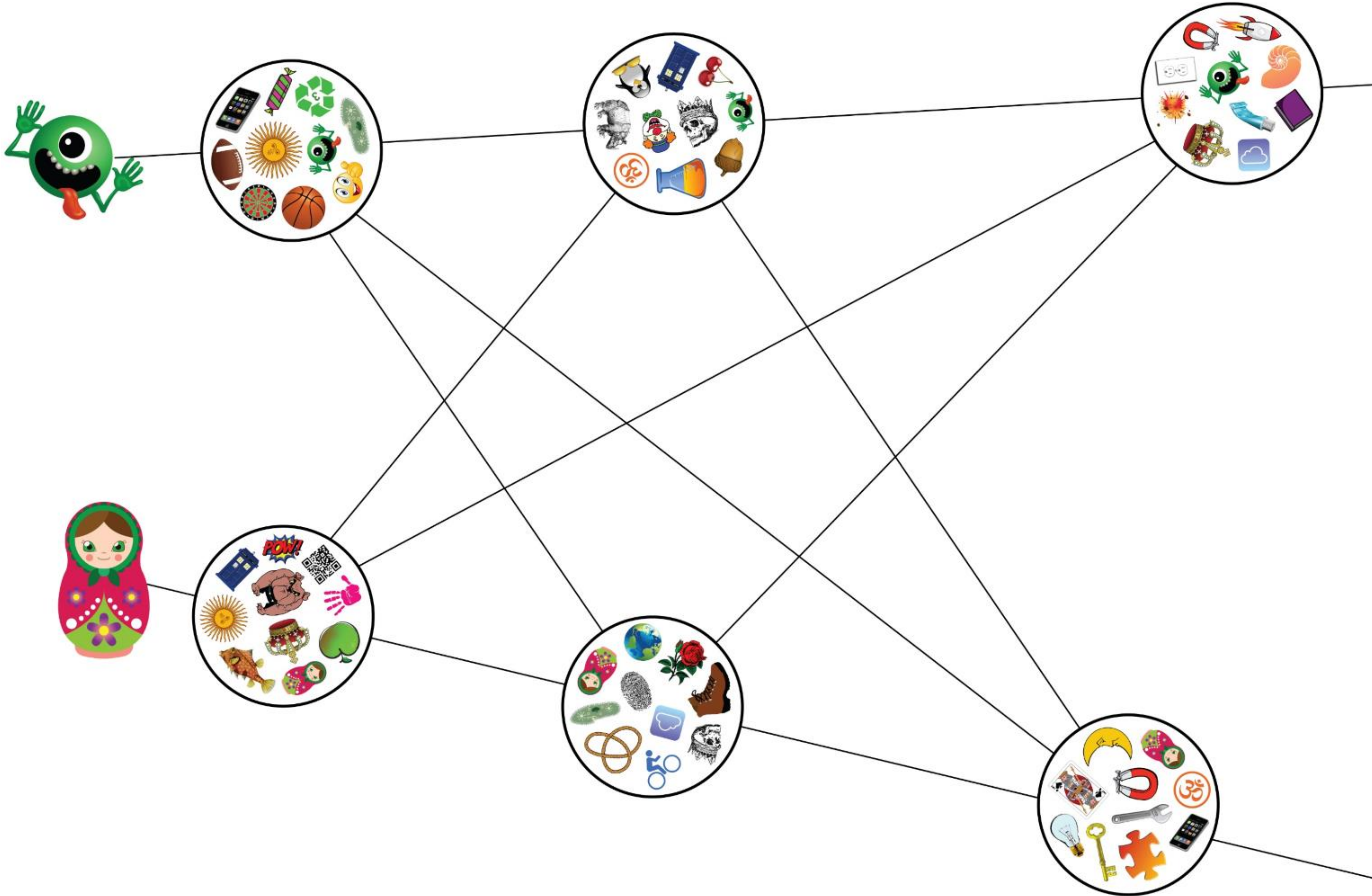



POW!


Omit


POW!


POW!


POW!


POW!


UNIVERSITY OF WYOMING
LEAVE NO DOUBT
icicle
goblin
slope 2
slope 1
dog
cactus
x
y

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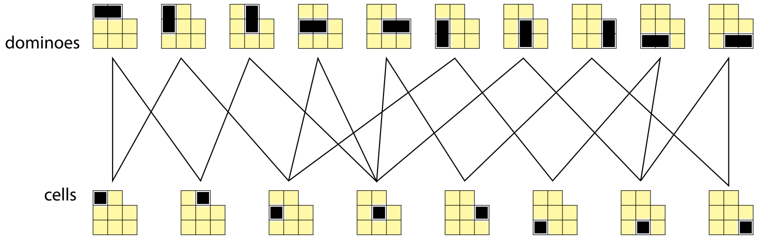
Consider a bipartite graph representing incidences between *points* and *blocks*.



A spread is a set of blocks partitioning the points.

Dually, an ovoid is a set of points partitioning the blocks.

Ovoids and Spreads

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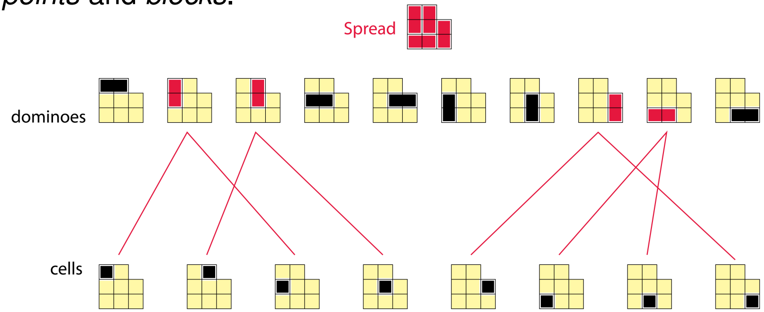

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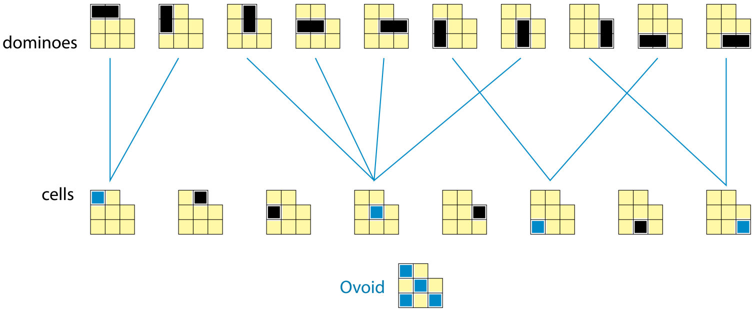

A spread is a set of blocks partitioning the points.

Dually, an ovoid is a set of points partitioning the blocks.

The $O_4^+(q)$ quadric (hyperbolic quadric in projective 3-space) is a $(q+1) \times (q+1)$ grid.

- q+1 lines
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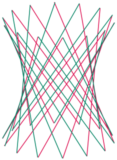

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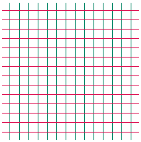
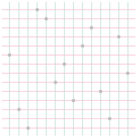


$(q+1)^2$ points;
$2(q+1)$ lines

It has $(q+1)!$ ovoids (and 2 spreads). Each ovoid is a 'transversal' of the grid, having $q+1$ points.



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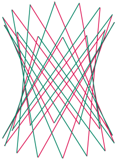

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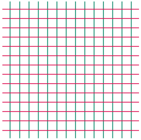

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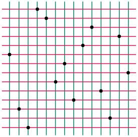

Ovoids in $O_6^+(q)$ (Klein Quadric)

The $O_6^+(q)$ quadric (Klein quadric) has

$(q^2+q+1)(q^2+1)$ points

$(q^2+1)(q+1)$ type I planes

$(q^2+1)(q+1)$ type II planes

Each ovoid has size $|\mathcal{O}| = q^2+1$ (same as a set of q^2+1 points of the quadric, no two perpendicular).

Ovoids in $O_6^+(q)$ are known to exist in great abundance.

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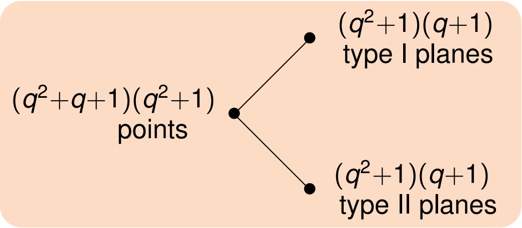

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Some ovoids in the Klein quadric $O_6^+(p)$

Consider a prime $p \equiv 1 \bmod 4$. Let $\mathcal{S}$ be the set of all $x = (x_1, \ldots, x_6) \in \mathbb{Z}^6$ such that

1. $x_i \equiv 1 \bmod 4$; and
2. $\sum_i x_i^2 = 6p$.

Then $|\mathcal{S}| = p^2 + 1$; and for all $x \neq y$ in $\mathcal{S}$, $x \cdot y \not\equiv 0 \bmod p$.

Example ($p = 5$, $|\mathcal{S}| = 5^2 + 1 = 26$)

$\mathcal{S}$ contains 6 vectors of shape $(5, 1, 1, 1, 1, 1)$;
20 vectors of shape $(-3, -3, -3, 1, 1, 1)$.

Example ($p = 13$, $|\mathcal{S}| = 13^2 + 1 = 170$)

$\mathcal{S}$ contains 20 vectors of shape $(5, 5, 5, 1, 1, 1)$;
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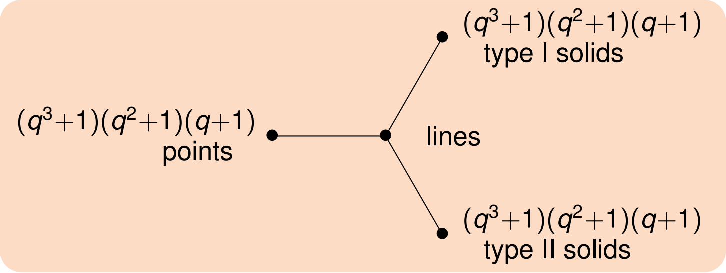

Each ovoid has size $|\mathcal{O}| = q^3 + 1$ (same as a set of $q^3 + 1$ points of the quadric, no two perpendicular).

Ovoids are equivalent to spreads (via triality).

When do they exist?

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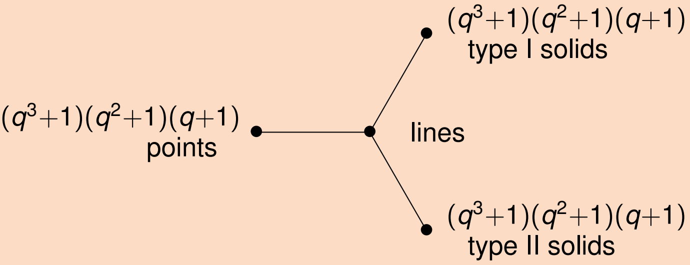

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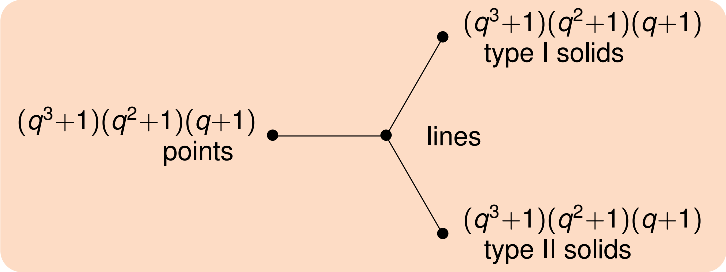

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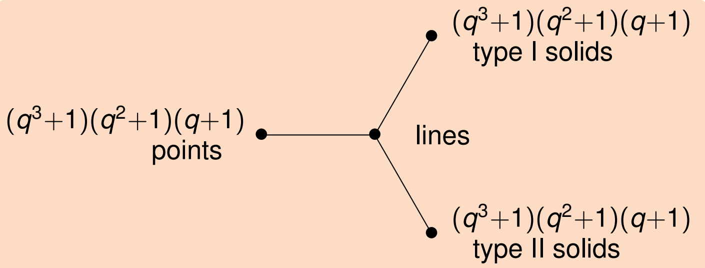

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The E_8 Root Lattice

Let E be the set of all vectors $\frac{1}{2}(x_1, x_2, \ldots, x_8) \in \mathbb{Q}^8$ such that

$$x_i \in \mathbb{Z}, \quad x_1 \equiv x_2 \equiv \cdots \equiv x_8 \mod 2, \quad \text{and } \sum_i x_i \equiv 0 \mod 4.$$

This is the E_8 root lattice. It is

- a lattice (i.e. discrete additive subgroup of $\mathbb{R}^8$);
- integral ($x \cdot y \in \mathbb{Z}$ for all $x, y \in E$);
- unimodular (its density is 1, i.e. it has one point per unit volume on average);
- it has minimum distance $\sqrt{2}$ (so for any $x \neq y$ in E, $\|y - x\| \geqslant \sqrt{2}$); and
- it is unique with these properties. Any subset of $\mathbb{R}^8$ of density 1 has minimum distance at most $\sqrt{2}$; and up to isometry, E is the unique subset attaining this optimum.

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E has 240 shortest vectors ($e \in E$, $\|e\|^2 = e \cdot e = 2$) called root vectors:

- $(\pm1, \pm1, 0, 0, 0, 0, 0, 0)$ and permutations thereof (112 vectors of this shape); and
- $\frac{1}{2}(\pm1, \pm1, \ldots, \pm1)$ with an even number of '$-$' signs (128 vectors of this shape).

For an odd prime p, there are $240(p^3+1)$ vectors $x \in E$ with $\|x\|^2 = 2p$.

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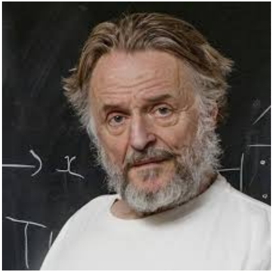
Theorem (Conway et al., 1988)

For every prime p, there is an ovoid in the $O_8^+(p)$ triality quadric.

Take p to be an *odd* prime (the case $p = 2$ was previously solved). Fix a root vector $e \in E$. Let $\mathcal{S}$ be the set of all $v \in E$ such that $\|v\|^2 = 2p$ and $v \in e + 2E$. We easily conclude that $|\mathcal{S}| = 2(p^3+1)$ and $\mathcal{S}$ consists of p^3+1 pairs $\pm v$ which reduce (mod p) to give an ovoid in the triality quadric. □



John H. Conway

Ovoids of Conway, Kleidman and Wilson (1988)

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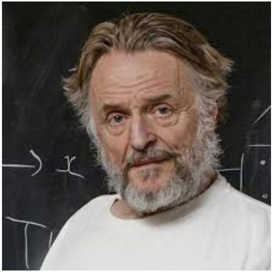

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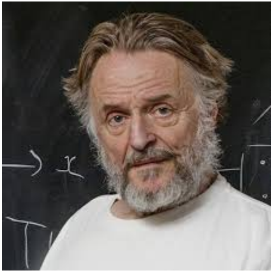

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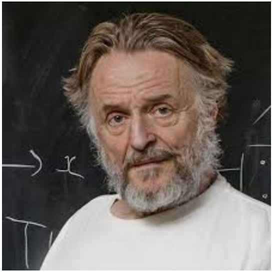

John H. Conway

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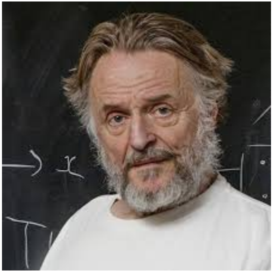

John H. Conway

More E_8-type ovoids in $O_8^+(p)$

We generalized Conway's construction (M., 1993) to a larger class of ovoids in $O_8^+(p)$, denoted

$$\mathcal{O}_{r,p}(u)$$

where $r \neq p$ are primes, $u \in E$ such that $\left(\dfrac{-p\|u\|^2/2}{r}\right) = +1$.

(The cases $r = 2, 3$ are in the original Conway paper.)

$\mathcal{O}_{r,p}(u)$ is formed using vectors $x \in \mathbb{Z}u + rE \subset E$ of norm $\|x\|^2 = 2k(r-k)p,\ \ 1 \leqslant k \leqslant \lfloor \frac{r-1}{2} \rfloor$.

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How many E_8-type ovoids are there?

Denote by n_p the number of equivalence types of ovoids (up to isometry and similarity) arising from our construction.

We conjectured that $n_p \to \infty$ as $p \to \infty$:

p	2	3	5	7	11	13	17	19	23
n_p	1	1	2	2	4	4	7	6	10

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Counting Ovoids

So instead we count the *total* number of ovoids

$$N_p = \sum_{i=1}^{n_p} [G : G_{\mathcal{O}_i}]$$

where $\mathcal{O}_i$ are representatives of the n_p equivalence types under G, the full group of isometries and similarities.

Conjecture

For $p > 3$, $N_p = [G : W]\frac{p^4-1}{2}$.

Here $|G| = 2p^{12}(p^6-1)(p^4-1)^2(p^2-1)$, $W = W(E_8)/\{\pm I\}$, $|W| = 348{,}364{,}800$. This formula may be rewritten in the more convenient form

$$\sum_{i=1}^{n_p} \frac{|W|}{|G_{\mathcal{O}_i}|} = \sum_{i=1}^{n_p} \frac{[W : W_{\mathcal{O}_i}]}{[G_{\mathcal{O}_i} : W_{\mathcal{O}_i}]} = \frac{p^4-1}{2}$$

which we refer to as the Conjectured Mass Formula.

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is strongly supported by the following table of values:

p	n_p	Mass Formula
5	2	$120+192 = 312 = \frac{5^4-1}{2}$
7	2	$120+1080 = 1200 = \frac{7^4-1}{2}$
11	4	$120+240+1920+5040 = 7320 = \frac{11^4-1}{2}$
13	4	$120+2160+3360+8640 = 14280 = \frac{13^4-1}{2}$
17	7	$120+240+1080+1920+6720+8640+23040 = 41760 = \frac{17^4-1}{2}$
19	6	$120+240+2160+15120+17280+30240 = 65160 = \frac{19^4-1}{2}$
23	10	$120+240+240+1080+1920+5040+6720$ $+15120+40320+69120 = 139920 = \frac{23^4-1}{2}$

The exceptional case $p = 3$

Our conjectured mass formula fails when $p = 3$ since in this case alone, the ovoids lie in an $O_7(p)$ hyperplane.

See Ball, Govaerts and Storme (2006).

Lower bounds for n_p

If the Conjectured Mass Formula holds, then trivially

$$[G:W]^{\frac{p^4-1}{2}} = N_p = \sum_{i=1}^{n_p} [G:G_{\mathcal{O}_i}] \leqslant n_p|G| \quad \Rightarrow \quad n_p \geqslant Cp^4$$

as $p \to \infty$. This estimate is conservative since *most* of the E_8-type ovoids have $|G_{\mathcal{O}}| \ll |G|$.

Exercise: Find an ovoid with $G_{\mathcal{O}} = 1$.

Show that $G_{\mathcal{O}} = 1$ for most E_8-type ovoids.

What are reasonable *upper* bounds for $|G_{\mathcal{O}}|$?

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The problem for general q

This construction fails for $q = p^r$, $r > 1$. Why?

It is natural to extend $\mathbb{Z} \subset A$, the ring of integers in a number field, such that $A/pA \cong \mathbb{F}_q$.

Also $E \subset \widehat{E} = E \otimes_{\mathbb{Z}} A$, $\widehat{E}/p\widehat{E} \cong \mathbb{F}_q^8$.

However, counting vectors of fixed norm in $\widehat{E}$ does not produce the necessary numbers for ovoids in $O_8^+(q)$.

Bigger problem: ovoids with the right automorphism groups apparently do not exist unless $q = p$. Why is this?

Compare: Our S_6-invariant ovoids in $O_6^+(q)$ apparently do not exist unless $q = p$. Why is this?

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\begin{tikzcd}
\text{E_8 root lattice} & E \arrow[d, "\text{mod } p"'] & O \arrow[l, "\text{forget multiplication}"'] \arrow[d, "\text{mod } p"] & \text{ring of integral octaves} \\
\text{orthogonal space} & (\mathbb{F}_p^8, Q) & \mathfrak{C}_p \arrow[l, "\text{forget multiplication}"] & \text{Cayley algebra}
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There are are essentially $[G : W] = O(p^{28})$ choices of 'E_8 structure' that can be imposed on the orthogonal space $(\mathbb{F}_p^8, Q)$, but $p^6(p^4-1)^2 = O(p^{14})$ choices of Cayley algebra structure.

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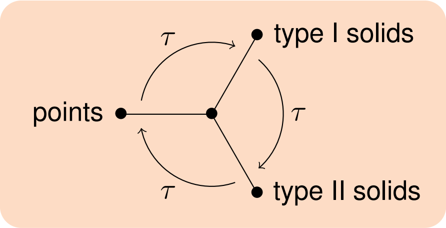
Triality automorphisms via the Cayley algebra $\mathfrak{C}_q$

$\langle x \rangle$



$$\begin{aligned}\langle x \rangle^{\tau} &= \mathfrak{C}_q x \\ &= \{y \in \mathfrak{C}_q \;:\; yx^* = 0\}\end{aligned}$$

$$\begin{aligned}\langle x \rangle^{\tau^2} &= x\mathfrak{C}_q \\ &= \{y \in \mathfrak{C}_q \;:\; x^* y = 0\}\end{aligned}$$

The role of nonassociative algebra

An alternative description of Conway's 'binary' ovoids $\mathcal{O} = \mathcal{O}_{2,p}(u)$, $p > 2$, $u \in O^{\times} = \{\text{roots of } E\}$:

In the ring O of integral octaves, the element $p \in O$ has $240(p^3 + 1)$ factorizations into irreducibles as $p = x^*x$, $x \in O$.

If we restrict $x \in e + 2O$ then there are $p^3 + 1$ pairs $\{\pm x\}$ of irreducibles. These give an ovoid in $O/pO \simeq \mathbb{F}_p^8$.

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Other ovoid constructions from E_8

In our 1993 construction, the sublattice $rE \subset E$ can be replaced by $wO \subset O$ where $w \in O$.

Unfortunately (?) the resulting ovoids are not new.

The sublattice $wO \subset O$ is not an ideal of O due to nonassociativity (i.e. wO it is not a right O-module). Indeed, every right or left ideal of O is a two-sided ideal rO where $r \in \mathbb{Z}$.

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So $\mathfrak{C}_q$ is a (nonassociative) alternative algebra of dimension 8 over $\mathbb{F}_q$. The units of $\mathfrak{C}_q$ form a Moufang loop $\mathfrak{C}_q^\times$ of order $q^3(q^4-1)(q-1)$.

Fix your favourite ovoid $\mathcal{O}$ in $O_8^+(q)$. We view $\mathcal{O}$ as a set of q^3+1 zero divisors in $\mathfrak{C}_q$, no two perpendicular.

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These bijections appear (but perhaps not so explicitly) in my ovoid construction. But . . .

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Bijections between ovoids

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Connections to the Cayley plane

It is possible to make some sense of the E_8-type construction by embedding these ovoids as objects in the Cayley plane: vectors in O of norm $k(r-k)$ are factorizable as xy where $x^*x = r$ and $y^*y = r-k$. Such pairs $(x,y) \in O^2$ arise naturally in the rational Cayley plane from intersections of lines with certain 'conics'. Details?

This suggests a bigger (possibly related) question:

We know that there cannot be quadrics in $O_{24}^{+}(q)$, at least for any reasonably small q.

But what are the right geometric objects of interest in L/pL where L is the Leech lattice?

Connections to the Cayley plane

It is possible to make some sense of the E_8-type construction by embedding these ovoids as objects in the Cayley plane: vectors in O of norm $k(r-k)$ are factorizable as xy where $x^*x = r$ and $y^*y = r-k$. Such pairs $(x,y) \in O^2$ arise naturally in the rational Cayley plane from intersections of lines with certain 'conics'. Details?

This suggests a bigger (possibly related) question:

We know that there cannot be quadrics in $O^+_{24}(q)$, at least for any reasonably small q.

But what are the right geometric objects of interest in L/pL where L is the Leech lattice?

There exist ovoids in quadrics (including Klein and triality quadrics) over other rings such as $\mathbb{Z}/m\mathbb{Z}$. Some are constructible from E_8.

What do we make of these?

How about ovoids over Galois rings?

Ovoids over other rings

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Thank You!

Questions?