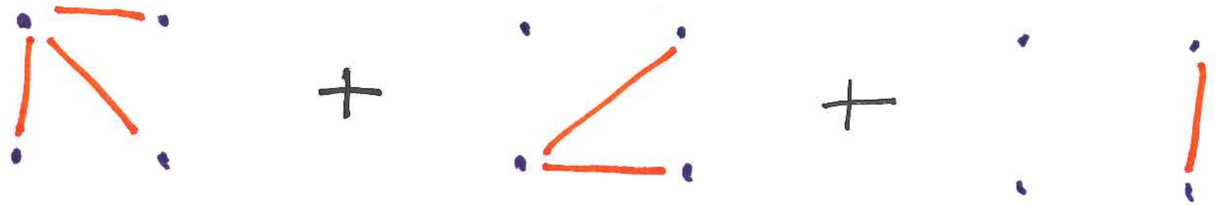
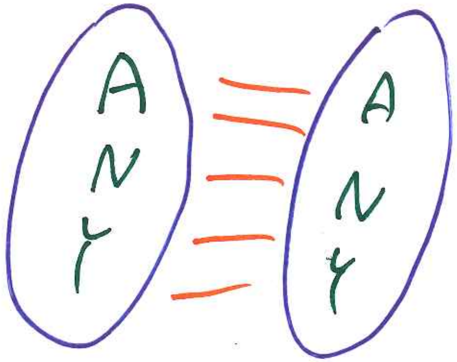
WITH LUKA MILICEVIC
AND TA SHENG TAN

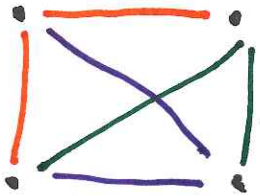
DECOMPOSE K_n INTO COMPLETE BIPARTITES - HOW MANY DO WE NEED?



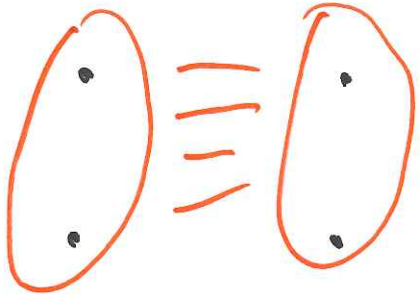


K_a K_b

$$1 + (a-1) + (b-1) = n-1$$



GRAHAM-POLLAK (1971): DO NEED $n-1$

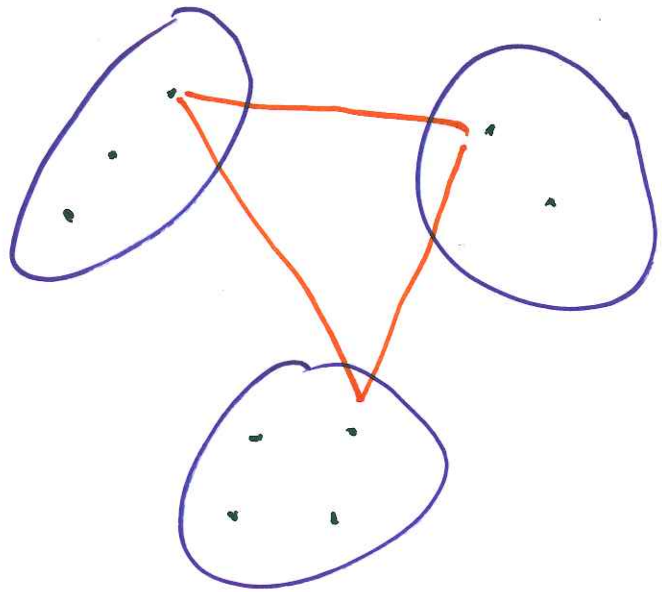
 $\longleftrightarrow (X_1+X_3)(X_2+X_4)$

$L_1R_1+\dots+L_{n-2}R_{n-2} = \left(\sum X_i\right)^2 - \sum X_i^2$ (TIMES $\frac{1}{2}$)

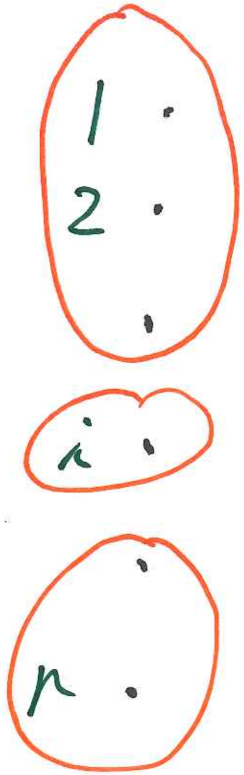
PUT $L_1=\dots=L_{n-2}=\sum X_i=0$: #

r-GRAPHS? SAY r = 3:

WANT TO DECOMPOSE COMPLETE 3-GRAPH INTO COMPLETE 3-PARTITE 3-GRAPHS

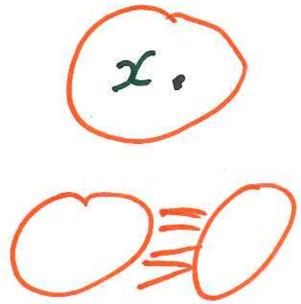


2.3.4 TRIPLES



= ALL 3-SETS WITH MIDDLE TERM i

ACHIEVES $n-2$



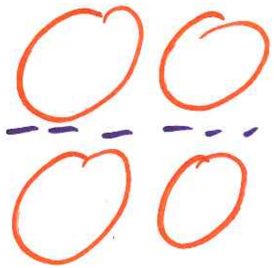
FIX x, AND LOOK AT THE PIECES INVOLVING x: WE SEE A DECOMPOSITION OF K_{n-1} INTO COMPLETE BIPARTITES. SO $\geq n-2$ PIECES

$r=4$:

= ALL 4-SETS WITH 2ND TERM i AND 4TH TERM j

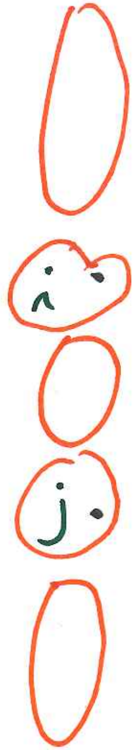
ACHIEVES $\binom{n-2}{2}$

<u>ALON</u> (1986): NEED $\geq \frac{1}{3}\binom{n-2}{2}$



+ 2 MORE

$r = 5$:



= ALL 5-SETS WITH 2ND TERM i AND 4TH TERM j

ACHIEVES $\binom{n-3}{2}$

CAN ACHIEVE $\binom{n}{r/2}$ FOR r EVEN

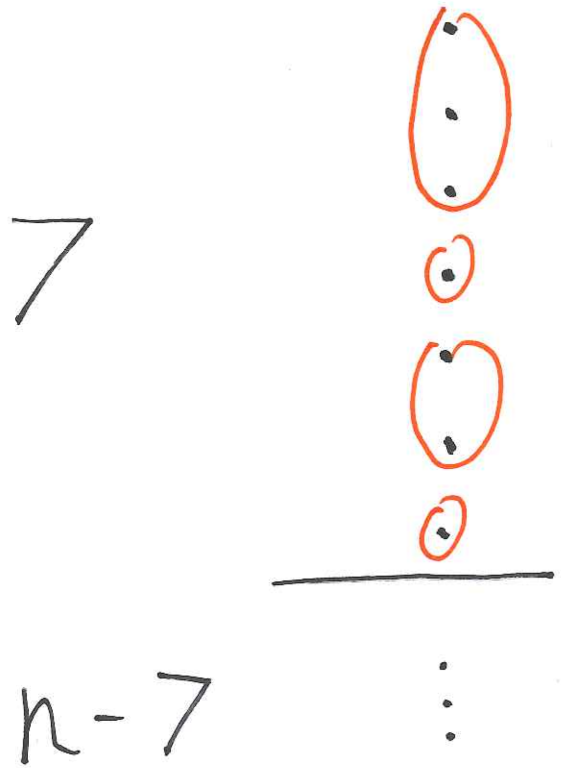
$\binom{n}{\frac{r-1}{2}}$ FOR r ODD

(TIMES $1+o(1)$)

IS THIS CONSTRUCTION (ASYMPTOTICALLY) OPTIMAL ?

NOT EXACTLY OPTIMAL – E.G. $r=4$:

$f(7) = 9$ WHEREAS $\binom{7-2}{2} = 10$



REPLACE THESE 10 BY THE 9

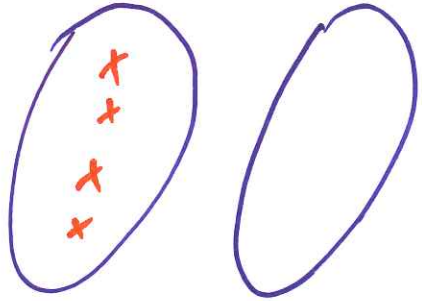
SO $f(n) \leq \binom{n-2}{2} - 1$ FOR $n \geq 7$

LOWER-ORDER

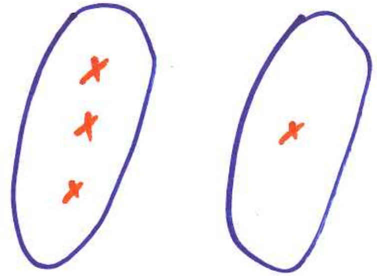
<u>CIOABA & TAIT</u> (2013): GAVE A SIMILAR LOWER-ORDER IMPROVEMENT TO $f_r(n)$, $r \geq 4$

<u>CIOABA, KUNDGEN & VERSTRAETE</u> (2009): GAVE A LOWER-ORDER IMPROVEMENT TO ALON'S LOWER BOUND

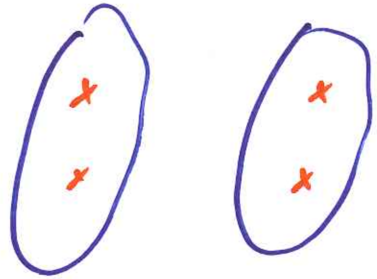
$$f(2n) \leq 2f(n)$$



$$+\, 2(n-2)$$



$$+\, g(n)$$



DECOMPOSE $E(K_n) \times E(K_n)$ INTO BLOCKS, MEANING (COMPLETE BIP.) $\times$ (COMPLETE BIP.)

CERTAINLY $g(n) \le (n-1)^2$.

IF $g(n) \le \lambda n^2$, WHERE $\lambda < 1$,

THEN FROM $f(2n) \le 2f(n) + g(n)$

$$4\alpha n^2 \qquad 2\alpha n^2 \qquad \lambda n^2$$

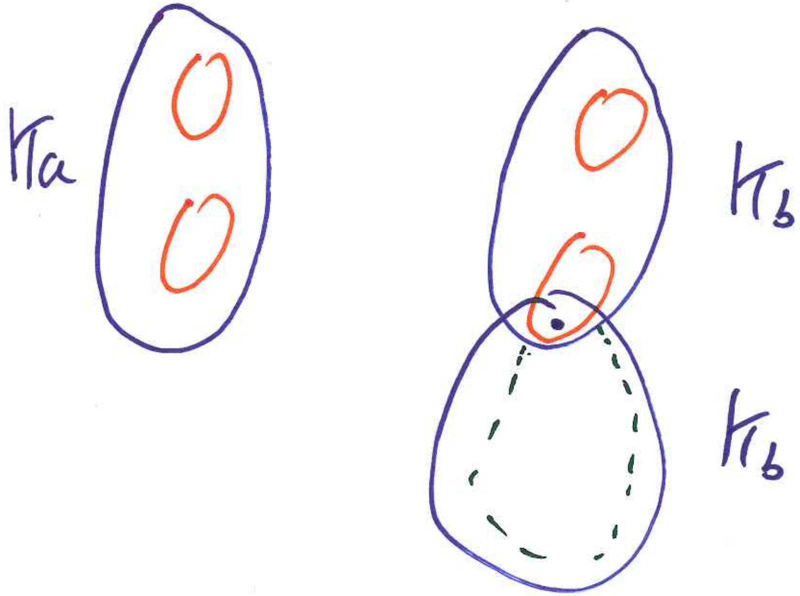
WE GET

$f(n) \le \alpha n^2$ WHERE $\alpha = \lambda/2$.

SO WHAT?

IF $g(a,b) = \lambda(a-1)(b-1)$ FOR SOME a,b $(\lambda < 1)$

THEN



$$g(a, 2b-1) \leq \lambda(a-1)(b-1) + \lambda(a-1)(b-1)$$
$$= \lambda(a-1)(2b-1-1)$$

KEEP GOING: $g(n) \leq \lambda(n-1)^2$ (TIMES $1+o(1)$)

SO <u>ONE</u> EXAMPLE OF $g(a,b) < (a-1)(b-1)$

SHOWS $f(n)$ IS <u>NOT</u> ASYMPTOTICALLY $\binom{n}{2}$

? $g(3,n) < 2(n-1)$?

LINEAR ALGEBRA $\Rightarrow$ $g(3,n) \geq \frac{9}{5}(n-1)$

? $g(4,n) < 3(n-1)$?

LINEAR ALGEBRA $\Rightarrow$ UNDER SOME ASSUMPTIONS THAT $g(4,n) \geq \frac{12}{5}(n-1)$

IN FACT $E(K_4) \times E(K_6)$ CAN BE DECOMPOSED INTO 14 BLOCKS

SO $g(n) \leq \frac{14}{15} n^2$

SO $f(n) \leq \frac{14}{15} \binom{n}{2}$

SAME FOR ANY EVEN $r \geq 4$: $f_r(n) \leq \frac{14}{15} \binom{n}{r/2}$

CONJECTURE: $g(n) = \frac{4}{5} n^2$ (TIMES $1 + o(1)$)

CONJECTURE: $f(n) = \frac{4}{5} \binom{n}{2}$

BACK TO GENERAL r: $f_r(n) \le \underbrace{\frac{14}{15}}_{\text{DOES NOT} \to 0} \binom{n}{r/2}$ (r EVEN)

TO MAKE THE COEFFICIENT $\to 0$, WE'D NEED $f_r(n) \le \lambda \binom{n}{\frac{r-1}{2}}$ FOR SOME ODD r

HARDER, AS CONSTRUCTION BETTER FOR r ODD

RECENTLY: $f_{295}(n) \leq \lambda \binom{n}{147}$, WHERE $\lambda < 1$

SO $f_r(n) \leq c_r \binom{n}{r/2}$ WITH $c_r \to 0$

QUESTION: HOW FAST DOES $c_r \to 0$?

QUESTION: DOES $f_5(n) = \binom{n}{2}$ (TIMES $1+o(1)$)