Generating Sets of Polar Grassmannians

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University of Delaware - August 21-24, 2019

Outline

- Basics on polar grassmannians

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- Generating sets and embeddings of polar grassmannians

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- Generating sets and embeddings of polar grassmannians
- Open problems

Polar spaces

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$$X^{\perp} := \{x \in \mathcal{P} \colon x \perp y, \ \forall y \in X\}, \ X \subseteq \mathcal{P}$$

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✓ The *rank of* Δ ($\mathrm{rank}(\Delta)$) is the vector dimension of the maximal singular subspaces of Δ.

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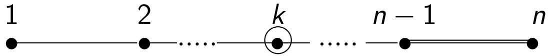
Definition

The *anisotropic defect* of Δ $(\mathrm{def}(\Delta))$ is the least upper bound of the lengths of the well ordered chains of $\mathfrak{N}(\Delta)$ w.r.t. inclusion.

Polar grassmannians

Δ: non-degenerate polar space of rank n



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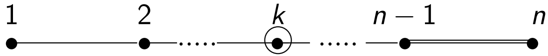

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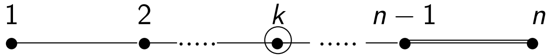

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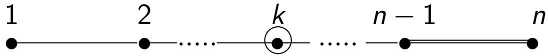

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* Lines of Δ_k:
  $k < n$: sets $\ell_{X,Y} := \{Z \colon X < Z < Y\}$, with $\mathrm{dim}(X) = k-1$, $\mathrm{dim}(Y) = k+1$ and Y singular subspace.

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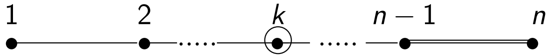

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$k = n$: sets $\ell_X := \{Z \colon X < Z\}$ with $\dim(X) = n-1$ and Z singular subspace.

Δ_1 : polar space; Δ_n : dual polar space.

Generation

$\Delta_k = (\mathcal{P}_k, \mathcal{L}_k)$: k–polar grassmannian

$S \subseteq \mathcal{P}_k$

Span of S: $\langle S \rangle_{\Delta_k} := \cap\{X : X \supseteq S, X \text{ subspace of } \Delta_k\}$

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Definition

1. $S \subseteq \mathcal{P}_k$ is a *generating set* of Δ_k if $\langle S \rangle_{\Delta_k} = \mathcal{P}_k$.
2. The *generating rank* of Δ_k is $\mathrm{gr}(\Delta_k) := min\{|S| \colon S \text{ is a generating set of } \Delta_k\}$.

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Polar grassmannians* admit the universal embedding.

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Definition

The *embedding rank* of Δ_k is $\mathrm{er}(\Delta_k) := \dim(\varepsilon^{univ})$.

$\varepsilon : \Delta_k \to \mathrm{PG}(V)$: projective embedding of Δ_k

S: generating set of Δ_k

$\downarrow$

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$$\dim(\varepsilon) \leq \mathrm{gr}(\Delta_k).$$

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$$\dim(\varepsilon) \leq \mathrm{gr}(\Delta_k).$$

$$\downarrow$$

$$\boxed{\mathrm{er}(\Delta_k) \leq \mathrm{gr}(\Delta_k)}$$

* If ε is a projective embedding of Δ_k with $\dim(\varepsilon) = \mathrm{gr}(\Delta_k)$ then ε is the universal embedding of Δ_k.

$\Delta_k = (\mathcal{P}_k, \mathcal{L}_k)$ polar grassmannian

Generation of Δ_k	Embeddings of Δ_k
* generating set ?	* Dimension ?
* generating rank ?	* Universality ?
	* Transparency ?
	* Application: codes

Polar spaces embedded

Δ: non-degenerate polar space of $\mathrm{rank}\,(\Delta) = n$ and $\mathrm{def}(\Delta) = d$.
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$*$ $\varepsilon(\Delta)$ is associated to a non-degenerate alternating, Hermitian or quadratic form f of V

Δ	$\leftrightarrow$	$\Delta(f)$
singular subspaces of Δ	$\leftrightarrow$	totally f-singular subspaces of V
$\mathrm{rank}(\Delta)$	$=$	Witt index of f

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* If $\mathrm{char}(\mathbb{F}) = 2$ then f cannot be alternating.
* If $\mathrm{char}(\mathbb{F}) \neq 2$ or f is Hermitian then ε is the unique embedding of Δ.
* If $\mathrm{char}(\mathbb{F}) = 2$ and f is quadratic then ε admits several proper quotients associated to *generalized quadratic forms*.

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$$V = \underbrace{V_1 \oplus V_2 \oplus \cdots \oplus V_n}_{\text{mutually orthogonal hyperbolic 2-spaces}} \quad \oplus \quad \underbrace{V_0}_{\text{anisotropic space orthogonal to } V_i}$$

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mutually orthogonal hyperbolic 2-spaces — anisotropic space orthogonal to V_i

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Theorem [I.C., L. Giuzzi, A. Pasini, 2019]

$$\mathrm{def}(\Delta) = \mathrm{def}(f)(= \dim(V) - 2n)$$

$V = V(N, \mathbb{F})$
f: non-deg. Hermitian or quadratic form of Witt index $n > 2$.
$\Delta(f)$: non-degenerate polar space associated to f.

$$1 \leq k \leq n$$

Δ_k: polar k-Grassmannian associated to f.

f hermitian $\rightarrow$ $\mathcal{H}_k$: Hermitian k-grassmannian
$\mathcal{H}$: Hermitian polar space

f quadratic $\rightarrow$ $\mathcal{Q}_k$: Orthogonal k-grassmannian
$\mathcal{Q}$: Orthogonal polar space

$*$ The points of Δ_k are points of the k-projective Grassmannian $\mathcal{G}_k$ of $\mathrm{PG}(V)$ and for $k < n$ also the lines of Δ_k are lines of $\mathcal{G}_k$.

Hermitian grassmannians- known results at 2018.

$\mathcal{H}$: non-degenerate Hermitian polar space of rank n
$\mathcal{H}_k$: Hermitian grassmannian of rank n

$def(\mathcal{H}_k)$	k	$\mathbb{F}$	$\mathrm{gr}(\mathcal{H}_k)$	$\mathrm{er}(\mathcal{H}_k)$	References
$d = 0, 1$	1	any	$2n + d$	$2n + d$	folklore
0	> 1	$\neq \mathbb{F}_4$	$\binom{2n}{k}$	$\binom{2n}{k}$	[1998-2012]
0	n	$\mathbb{F}_4$	?	$(4^n + 2)/3$	[2002]
1	n	$\mathbb{F}_q$	$\leq 2^n$	—	[2001]

Orthogonal grassmannians- known results at 2018. 1/2

$\mathcal{Q}$: non-degenerate Orthogonal polar space of rank n

$\mathcal{Q}_k$: Orthogonal grassmannian of rank n

$def(\mathcal{Q}_k)$	k	$\mathbb{F}$	$\mathrm{gr}(\mathcal{Q}_k)$	$\mathrm{er}(\mathcal{Q}_k)$	References
$0 \leq d \leq 2$	1	*any*	$2n + d$	$2n + d$	folklore
0	n	*any*	$\prod_{i=0}^{n-1}(\lvert\mathbb{F}\rvert^i + 1)$	$---$	[1983]
$d = 1, 2$	n	$\mathrm{char}(\mathbb{F}) \neq 2$	2^n	2^n	[1998-2011]
2	n	$\mathrm{char}(\mathbb{F}) = 2$	2^n	2^n	[2001],[2011]

Orthogonal grassmannians- known results at 2018. 2/2

$def(\mathcal{Q}_k)$	k	$\mathbb{F}$	$\mathrm{gr}(\mathcal{Q}_k)$	$\mathrm{er}(\mathcal{Q}_k)$	Ref.
1	n	$\neq \mathbb{F}_2$	$\binom{2n}{n} - \binom{2n}{n-2}$	$\binom{2n}{n} - \binom{2n}{n-2}$	[2007]
1	n	$\mathbb{F}_2$	?	$\frac{(2^n+1)(2^{n-1}+1)}{2}$	[2001]
$d = 0, 1, 2$	$2 < n$	$\mathbb{F}_p$	$\binom{2n+d}{2}$	$\binom{2n+d}{2}$	[1998]
$d = 0, 1$	2	any	$\leq \binom{2n+d}{2} + g$	?	[2001]
$d = 0, 1$	$2, 3 < n$	*perfect*, $\mathrm{char}(\mathbb{F}) > 0$ *number field*	?	$\binom{2n+d}{k}$	[2013]

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New results - Extremal cases: $k = 1$ and $k = n$

Δ: non-degenerate polar space of $\mathrm{rank}\,(\Delta) = n$ and $\mathrm{def}(\Delta) = d$.
H: hyperplane of Δ
(i.e. H is a proper subspace s.t. $|H \cap \ell| = 1$ or $\ell \subseteq H$, $\forall \ell$ line of Δ)

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Theorem [I.C., L. Giuzzi, A. Pasini, 2019]

1. If $d > 0$ the set H_n of n-singular subspaces contained in H is a generating set of the dual polar space Δ_n.
2. $\mathrm{gr}(\Delta_n) \leq \mathrm{gr}(H_n)$.

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Theorem [I.C., L. Giuzzi, A. Pasini, 2019]

If there exists at least one maximal well ordered chain of nice subspaces of Δ and Δ admits an embedding of dimension $2n + d$, then $\mathrm{gr}(\Delta) = \mathrm{er}(\Delta) = 2n + d$.

Constructions for $2 \leq k < n$

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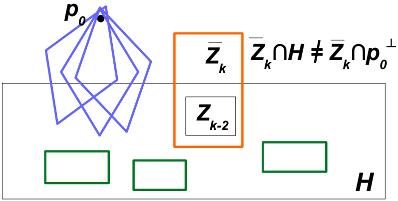
$\overline{G}_{H,p_0} := \{\overline{Z}_k : Z_{k-2} \in G_{H,p_0}\}$ where G_{H,p_0} is a generating set of $(H \cap p_0^{\perp})_{k-2}$.



$$S_k(H, p_0, \overline{G}_{H,p_0}) := S_k(H) \cup S_k(p_0) \cup \overline{G}_{H,p_0}$$

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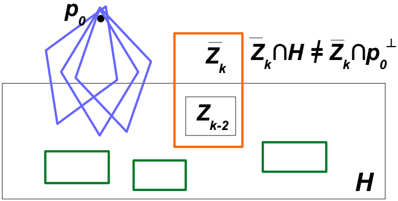

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Theorem [I.C, L.Giuzzi, A. Pasini, 2019]

The set $S_k(H, p_0, \overline{G}_{H,p_0})$ spans Δ_k for any $k = 2, 3, ..., n-1$.

Constructions for $2 \leq k < n$

Δ: non-degenerate polar space of $\mathrm{rank}\,(\Delta) = n$ and $\mathrm{def}(\Delta) = d$.
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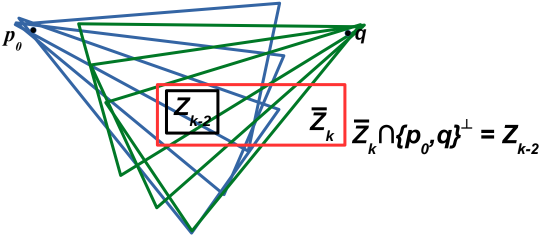

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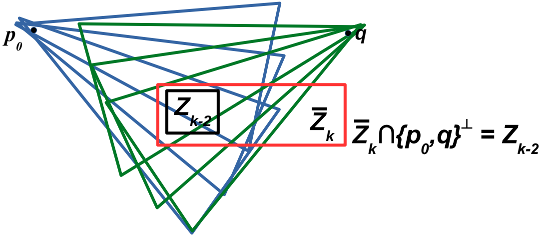

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Generation of Hermitian Grassmannians

$\mathcal{H}$: non-deg. Hermitian polar space of rank n and $def(\mathcal{H}) = d$.
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Corollary [I.C., L.Giuzzi, A. Pasini, 2019]

If $d < \infty$, $k < n$ (and $\mathbb{F} \neq \mathbb{F}_4$ when $k > 1$) then the Plücker embedding of $\mathcal{H}_k$ is universal.

Generation of Orthogonal Grassmannians

$\mathcal{Q}$: non-deg. orthogonal polar space of rank n and $def(\mathcal{Q}) = d$.
$\mathcal{Q}_n$: dual polar space of orthogonal type, $\mathbb{F}$: underlying field of $\mathcal{Q}$.

Theorem [I.C, L.Giuzzi, A. Pasini, 2019]

If $d > 0$ and $\mathrm{char}(\mathbb{F}) \neq 2$ then $\mathrm{gr}(\mathcal{Q}_n) \leq 2^n$.

When $0 < d \leq 2$ and $\mathrm{char}(\mathbb{F}) \neq 2$ the inequality $\mathrm{gr}(\mathcal{Q}_n) \leq 2^n$ is in fact an equality. Perhaps the same is true when $d > 2$ but, since $\mathcal{Q}_n$ is not (projectively) embeddable when $d > 2$, there is no easy way to prove it.

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* If *def*$(\mathcal{Q}(\mathbb{F}_p)) = 0, 1, 2$ then $\mathrm{gr}(\mathcal{Q}_2(\mathbb{F}_p)) = \binom{2n+d}{2}$, [Cooperstein, 1998].
* If *def*$(\mathcal{Q}(\mathbb{F})) = 0, 1$ and $[\mathbb{F}:\mathbb{F}_0] = g$ then $\mathrm{gr}(\mathcal{Q}_2(\mathbb{F})) \leq \binom{2n+d}{2} + g$, [Blok,Pasini, 2001].

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Theorem [I.C, L.Giuzzi, A. Pasini, 2019]

Let $\mathcal{Q}(\mathbb{F})$ be a non-degenerate orthogonal polar space of rank $n > 2$ in $\mathrm{PG}(2n + d - 1, \mathbb{F})$. Put $\mathbb{F} = \mathbb{F}_q$ with $q \in \{4, 8, 9\}$.

1. If $\left\{ \begin{array}{c} d = 0 \text{ and } n > 3 \\ \text{or} \\ d = 1, 2 \text{ and } n \geq 3 \end{array} \right\}$ then $\mathrm{gr}(\mathcal{Q}_2(\mathbb{F})) = \mathrm{er}(\mathcal{Q}_2(\mathbb{F})) = \binom{2n+d}{2}$.
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Theorem [I.C, L.Giuzzi, A. Pasini, 2019]

If $\mathcal{Q}(\mathbb{F})$ is a non-degenerate hyperbolic polar space of rank 3 in $\mathrm{PG}(5, \mathbb{F})$ then $\mathcal{Q}_2(\mathbb{F})$ is never $\mathbb{F}_0$-generated, for any proper subfield $\mathbb{F}_0$ of $\mathbb{F}$.

Hermitian grassmannians- known results till now.

$\mathcal{H}$: non-degenerate Hermitian polar space of rank n
$\mathcal{H}_k$: Hermitian grassmannian of rank n

$def(\mathcal{H}_k)$	k	$\mathbb{F}$	$\mathrm{gr}(\mathcal{H}_k)$	$\mathrm{er}(\mathcal{H}_k)$	References
0	1	*any*	$2n$	$2n$	folklore
0	> 1	$\neq \mathbb{F}_4$	$\binom{2n}{k}$	$\binom{2n}{k}$	[1998-2012]
0	n	$\mathbb{F}_4$	?	$(4^n + 2)/3$	[2002]
1	n	$\mathbb{F}_q$	$\leq 2^n$	—	[2001]
any	$\neq n$	$\neq \mathbb{F}_4$ if $k > 1$	$\binom{2n+d}{k}$	$\binom{2n+d}{k}$	[2019]
$d > 0$	n	$\neq \mathbb{F}_4$ if $k > 1$	$\leq 2^n$		[2019]

Orthogonal grassmannians- known results till now.

$def(\mathcal{Q}_k)$	k	$\mathbb{F}$	$\mathrm{gr}(\mathcal{Q}_k)$	$\mathrm{er}(\mathcal{Q}_k)$	Ref.
1	n	$\neq \mathbb{F}_2$	$\binom{2n}{n} - \binom{2n}{n-2}$	$\binom{2n}{n} - \binom{2n}{n-2}$	[2007]
1	n	$\mathbb{F}_2$	?	$\frac{(2^n+1)(2^{n-1}+1)}{2}$	[2001]
$d = 0, 1, 2$	$2 < n$	$\mathbb{F}_p$	$\binom{2n+d}{2}$	$\binom{2n+d}{2}$	[1998]
$d = 0, 1$	2	any	$\leq \binom{2n+d}{2} + g$	?	[2001]
$d = 0, 1$	$2, 3 < n$	*perfect*, $\mathrm{char}(\mathbb{F}) > 0$ *number field*	?	$\binom{2n+d}{k}$	[2013]
$d = 0$ *and* $n > 3$ $d = 1, 2$ *and* $n > 2$	$2 < n$	$\mathbb{F}_4, \mathbb{F}_8, \mathbb{F}_9$	$\binom{2n+d}{2}$	$\binom{2n+d}{2}$	[2019]
$d > 0$	n	$\mathrm{char}(\mathbb{F}) \neq 2$	$\leq 2^n$		[2019]

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- Generating rank of k-orthogonal grassmannians for $k > 2$.
- If $n > 3$, is $\mathcal{Q}^+_{n-1}(2n-1, \mathbb{F})$ generated over subfields?

Projective codes

Ω: set of N points of $\mathrm{PG}(V)$, $V = V(K, \mathbb{F}_q)$.

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Theorem

Any semilinear collineation of $\mathrm{P\Gamma L}(K, q)$ stabilizing Ω induces automorphisms of $\mathcal{C}(\Omega)$.

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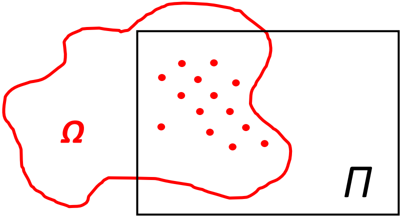

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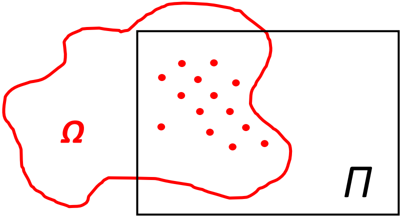

The study of the weights of $\mathcal{C}(\Omega)$ is equivalent to the study of the hyperplane sections of Ω.

Projective codes $\mathcal{C}(\Omega)$ associated to $\Omega = \varepsilon(\Theta)$.	$\hookrightarrow$	Embeddings of Θ: universal embedding and its quotient.	$\hookrightarrow$	Generating set of Θ

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$$N = \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}, \quad K = \binom{m}{k}, \quad d_{\min} = q^{(m-k)k}.$$

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$$N = \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}, \quad K = \binom{m}{k}, \quad d_{\min} = q^{(m-k)k}.$$

k–multilinear alternating forms on V $\leftrightarrow$ Hyperplanes of $\bigwedge^k V$

Grassmann Codes

$\mathcal{G}_k$: Grassmannian of the k-subspaces of $V(m,q)$.
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Remark

- *Minimum weight codewords in a Grassmann code correspond to non-null k–multilinear alternating forms with a maximum number of totally isotropic spaces.*
- *When $k = 2$ these are non–null alternating forms with maximum radical.*

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$\bar{\varepsilon}_k$: Grassmann embedding of $\bar{\Delta}_k$.

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$\mathcal{H}_k$: Hermitian Grassmannian of rank n and defect $d = 0, 1$
ε_k: Grassmann embedding of $\mathcal{H}_k$.

Theorem [Blok, Cooperstein, 2012; I.C., L. Giuzzi, A. Pasini, 2018]

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Definition

- Δ_k: Orthogonal/Hermitian/Symplectic grassmannian
- $\mathcal{C}(\Delta_k) := \mathcal{C}(\varepsilon_k(\Delta_k))$:

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Orthogonal Grassmann Codes

Theorem (I.C., L. Giuzzi, K.V. Kaipa, A. Pasini 2013–2017)

The known parameters of an Orthogonal Grassmann Code are

(n,k)	N	K	d
$1 \leq k < n$	$\prod_{i=0}^{k-1} \frac{q^{2(n-i)}-1}{q^{i+1}-1}$	$\binom{2n+1}{k}$	$d \geq \widetilde{d}(q,n,k)$
$(3,3)$	$(q^3+1)(q^2+1)(q+1)$	35	$q^2(q-1)(q^3-1)$
$(n,2)$	$\frac{(q^{2n}-1)(q^{2n-2}-1)}{(q-1)(q^2-1)}$	$(2n+1)n$	$q^{4n-5}-q^{3n-4}$

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$(3,3)$	$(q^3+1)(q^2+1)(q+1)$	28	$q^5(q-1)$
$(n,2)$	$\frac{(q^{2n}-1)(q^{2n-2}-1)}{(q-1)(q^2-1)}$	$(2n+1)n-1$	$q^{4n-5}-q^{3n-4}$

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$$\widetilde{d}(q,n,k) := (q+1)(q^{k(n-k)}-1)+1$$

Symplectic and Hermitian Grassmann codes

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Theorem (I.C., L.Giuzzi 2018)

The known parameters of a Hermitian Grassmann code are

$(n,2)$	N	K	d
$n=4,6$	$\frac{(q^n-1)(q^{n-1}+1)(q^{n-2}-1)(q^{n-3}+1)}{(q^2-1)^2(q^2+1)}$	$\binom{n}{2}$	$q^{4n-12}-q^{2n-6}$
$n\geq 8,$ *even*	$\frac{(q^n-1)(q^{n-1}+1)(q^{n-2}-1)(q^{n-3}+1)}{(q^2-1)^2(q^2+1)}$	$\binom{n}{2}$	q^{4n-12}
n *odd*	$\frac{(q^n+1)(q^{n-1}-1)(q^{n-2}+1)(q^{n-3}-1)}{(q^2-1)^2(q^2+1)}$	$\binom{n}{2}$	$q^{4n-12}-q^{3n-9}$

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- Implementation of $\mathcal{C}(\Delta_k)$, $k > 2$.