Erdős-Ko-Rado theorems in buildings

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In this talk we look at objects of a geometry and ask for

1. the largest number of objects no two of which are in general position,
2. the structure of the largest such sets.

The theorem of Erdős-Ko-Rado

Find the largest number of intersecting d-subsets from an n-set.

Point-Example. All d-sets containing a fixed element.

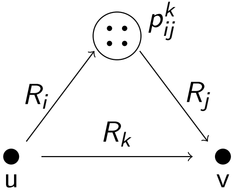
Theorem (Erdős-Ko-Rado, 1961)

If X is an intersecting family of d-subsets of an n-set with $n \geq 2d$, then $|X| \leq |Point\text{-}example|$. For $n \geq 2d+1$ equality holds if and only if X is the point-example.

Homogenous coherent configurations

Let Ω be a set and $R_1, \ldots, R_d$ be relations on Ω such that

- $R_1 = \{(u, u) \mid u \in \Omega\}$,
- Every pair (u, v) of $\Omega \times \Omega$ lies in exactly one relation.
- $R_i^\top \in \{R_1, \ldots, R_d\}$
- Regularity condition:



Bose-Mesner algebra

$\Omega = \{u_1, \ldots, u_n\}$, adjacency matrices $A_1, \ldots, A_d \in \mathbb{C}^{n \times n}$ defined by

$$A_k(i,j) := \begin{cases} 1 & \text{if } (u_i, u_j) \in R_k \\ 0 & \text{otherwise} \end{cases}$$

Then

- $A_1 = I_n$
- $A_1 + \cdots + A_d$ is the all-one matrix
- $A_i^\top \in \{A_1, \ldots, A_d\}$
- $A_i A_j = \sum_k p_{ij}^k A_k$

$\Rightarrow \quad \mathcal{A} := \langle A_1, \ldots, A_d \rangle_{\mathbb{C}}$ is $\mathbb{C}$-algebra. Usually it is not commutative.

The symmetric case

$A_i^\top = A_i \; \forall i$

$\Rightarrow$ $A_1, \ldots, A_d$ can be diagonalized simultaneously.

$\Rightarrow$ d comon eigenspaces $V_1, \ldots, V_d$.

$\Rightarrow$ Projections $E_1, \ldots, E_d$ on eigenspaces

$$A_j = \sum_{i=1}^{d} P_{ij} E_i$$

$$E_j = \sum_{i=1}^{d} Q_{ij} A_i$$

where $(P_{ij})(Q_{ij}) = I_n$. Regularity condition gives $Q_{ij} = \frac{P_{ji}}{|R_i|} m_j$

The linear programming bound

- Consider a subset X of $\Omega = \{u_1, \dots, u_n\}$
- Characteristic vector $\chi \in \mathbb{C}^n$: $\chi_i = 1$, if $u_i \in X$, and $\chi_i = 0$ if $u_i \notin X$.
- Distribution array $(x_1, \dots, x_d)$ of X with $x_i = \frac{1}{|X|}|R_i \cap (X \times X)|$

$$x_i = \frac{1}{|X|}\chi^\top A_i \chi \quad \text{and} \quad |X| = \sum x_i$$

- for $j = 1, \dots, d$

$$0 \leq \chi^\top E_j \chi = \sum_i Q_{ij} \chi^\top A_i \chi$$

$$\Rightarrow \quad 0 \leq \sum_{i=1}^{d} \frac{P_{ji}}{|R_i|} x_i$$

- No two of X in general position $\Rightarrow x_d = 0$.

Application: buildings of type A_n

Two k-subspaces U, U' of an n-dimensional vector space V of dimension $n \geq 2k$ are in general position, if $U \cap U' = \{0\}$.

Point-example: All k-subspaces containing a given 1-dim. subspace .

Theorem (Newman, 2004)

For $n \geq 2k$ every largest EKR-set is of this form.

Application: buildings of type C_n

Polar spaces other than hyperbolic quadrics

Two generators are in general position, if they are disjoint.

Point-Example: All generators containing a given point.

Theorem (Stanton, 1980)

Except when the polar space is of Hermitian type $H(2d-1, q^2)$, with $d \geq 3$ odd, the point example is a largest EKR-set.

Theorem (Pepe, Storme, Vanhove, 2011)

Classification of largest EKR-sets. Not all are point-examples.

The exception

Polar spaces $H(2d-1, q^2)$, $d \geq 3$, odd.

Example: $d = 3$. All generators (planes) meeting a given plane in at least a line. This is a largest EKR-set one (again Pepe et.al).

Point-example: $q^{(d-1)^2}$.

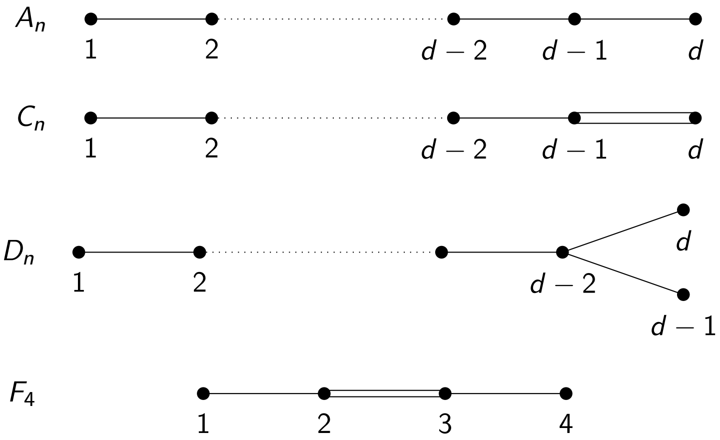
Hoffman bound $q^{(d-1)^2+(d-1)}$.

Theorem (Ihringer, M, 2014)

If X is an EKR set of generators of $H(2d-1, q^2)$, $d \geq 5$, odd, then

$$|X| \leq q^{(d-1)^2+1} + const \cdot q^{(d-1)^2}.$$

Some types of buildings



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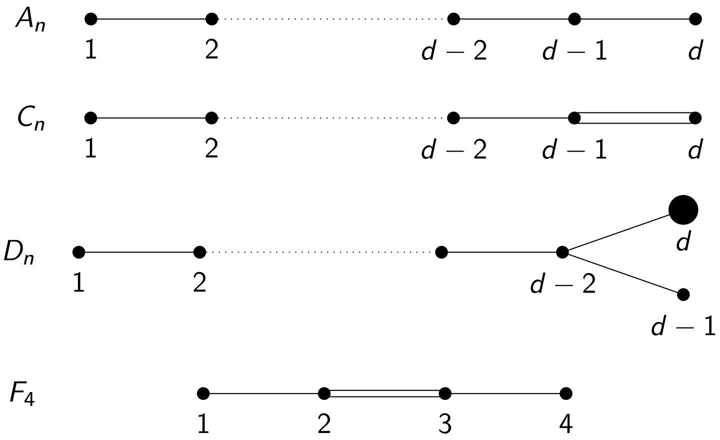

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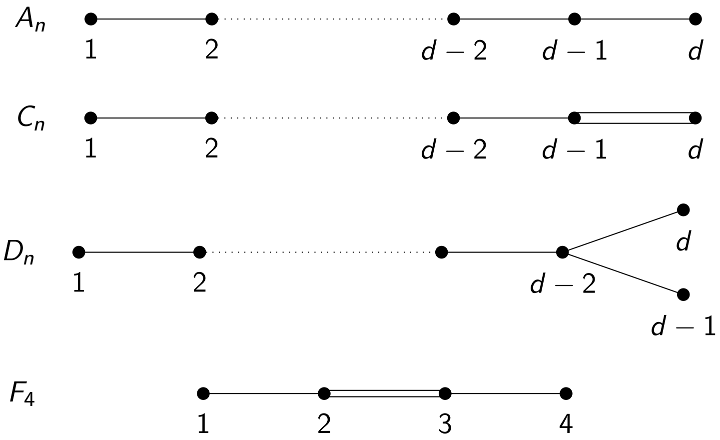
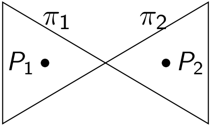


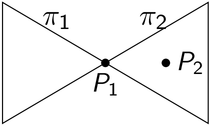
Some problems are trivial

Point-plane flags in PG(4, q)

General position:



Not symmetric:



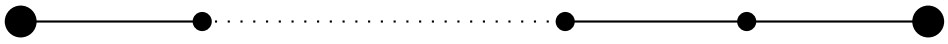
Example: Take solid S and all point-plane flags with its plane in S.

Theorem (Blokhuis, Brouwer, Szőnyi, 2014)

These are the largest EKR-sets

Point-Hyperplane flags in A_d

(P_1, H_1) and (P_2, H_2) are in general position iff $P_1 \notin H_2$ and $P_2 \notin H_1$.

A_d 

EXAMPLE: Take a chamber C of a projective space of rank d. Then

$$X := \{(P, H) \mid P \in S \subseteq H \text{ for some } S \in C\}$$

is an EKR-set of point-hyperplane flags.

Theorem (Blokhuis, Brouwer, Güven, 2014)

This is best possible and the only example of that size.

Lines in polar spaces

Two lines ℓ and h of a polar space are in general position iff $\ell^\perp \cap h = \emptyset$.

EXAMPLE: Let C be a chamber of the polar space. Then

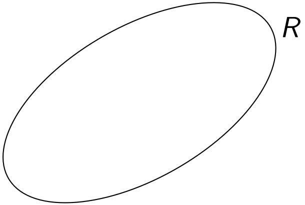
$$X \quad := \quad \{\ell \mid \ell \cap S \neq \emptyset,\ S \subseteq \ell^\perp \text{ for some } S \in C\}$$

is an EKR-Rado set of lines.

Theorem (M, 2019)

The above example is best possible for finite classical non-degenerate polar spaces of rank $d \geq 2$ and order $q > 2(d-1)$.

Lines in polar spaces



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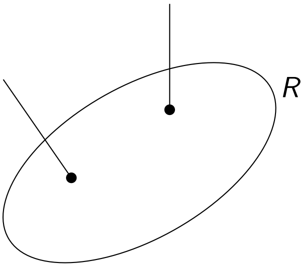

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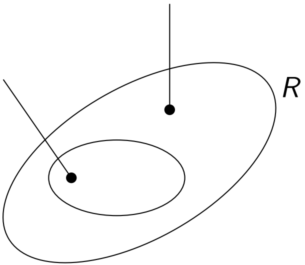

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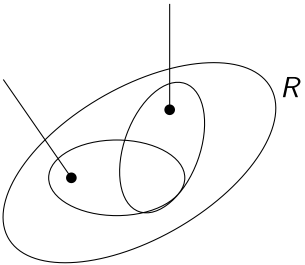


Point relations in F_4

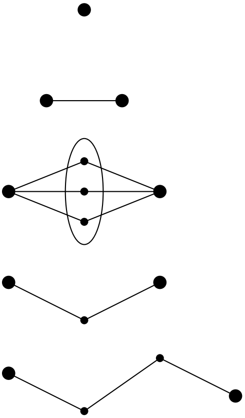
Two equal points

Collinear points

Symplectic pair

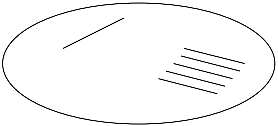
Special pair

Distance three



EKR sets of points in F_4

Choose S =Symplecton $F =$ EKR-set of lines of S.



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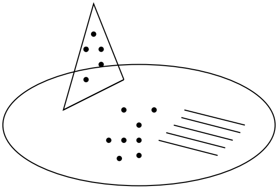
Define $X =$ set of all points of S, all points in planes on lines of F.



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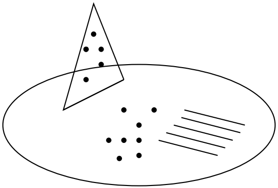

Alternative description with incident point-line pair (P, ℓ) (center)

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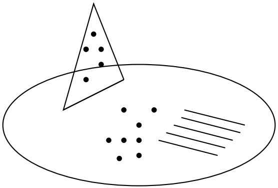

Alternative description with incident point-line pair (P, ℓ) (center)

Theorem (M, 2019)

The above example is best possible for all finite thick buildings of type F_4.

The general case: symmetric or not

Theory by Higman (1975, 1976, 1987)

- $A_i^\top \in \{A_1, \ldots, A_d\}$
- $\bar{\mathcal{A}}^\top = \mathcal{A}$.
- $\mathcal{A}$ is semisimple
- $\mathcal{A} \simeq \bigoplus_{i=1}^{r} \mathbb{C}^{s_i \times s_i}$
- Irreducible representations: $D_i : \mathcal{A} \to \mathbb{C}^{s_i \times s_i}$ with $D_i(A_j^*) = D_i(A_j)^*$

The result of Hobart

Let $X \subseteq \Omega$, $x_i = \frac{1}{n}\chi A_i^\top \chi$ and $C(X) := \sum_{i=1}^{d} \frac{x_i}{|R_i|} A_i$

Theorem (Hobart, 2009)

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For all j we have $Trace(D_j(C(X))) \geq 0$.

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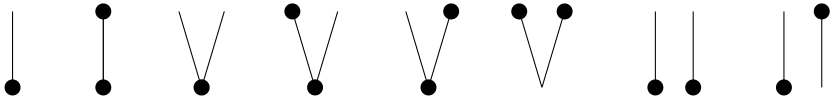
$Trace(D_j(C(X))$ can be calculated without knowing D_j!
Each representation gives a linear inequality in the parameters x_i of X.
Linear programming gives bound on $|X|$.

Example: Chambers in GQ's (type C_2)

Ω is set of chambers of a thick GQ of order (s, t).

There are eight relations:



Thus: $\dim(\mathcal{A}) = 8$ and $\mathcal{A}$ is not commutative. This implies that

$$\mathcal{A} \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^{2\times 2}$$

Example: Chambers in GQ's

The three linear inequalities (other than valency) of the 1-dimensional representation are sufficient for the correct bound $(s+1)(t+1)$.

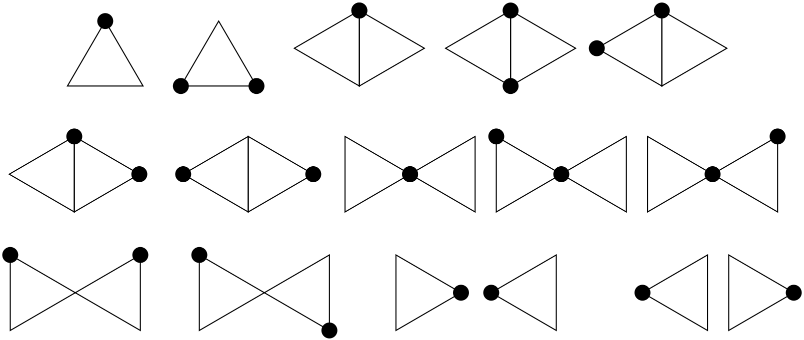
For $|X| = (s+1)(t+1) \Rightarrow$ information on distribution array $(x_1, \ldots, x_d)$.

$\Rightarrow$ Geometric classification is easy for order $(s,t) \neq (2,2)$.

For order $(s,t) = (2,2)$, there is a sporadic example in $Q(4,2)$, coming from an embedded $Q^+(3,2)$.

Example: Point-plane flags in ps of rank 3

There are 14 relations



Center has dimension 8, so $\mathcal{A} \cong 6\mathbb{C} \oplus 2\mathbb{C}^{2\times 2}$

Example: Point-plane flags in ps of rank 3

Theorem (M, 2019+)

An EKR-set X of point-plane flags in a polar spaces of rank 3 with $e \neq 0, \frac{1}{2}$ satisfies

$$|X| \leq (q^2 + q + 1)(q^e + 1)(q^{e+1} + 1)$$

.

This bound is sharp, one example attaining the bound consists of all point-plane flags with the point P in a given plane, a second examples consists of all point-plane flags with its plane on a given point.

Open problems

1. More examples?
2. What happens for $e = \frac{1}{2}$.

Example: chambers in A_3

EKR-sets of chambers (point-line-plane flags) (P, ℓ, E) in $\mathrm{PG}(3, q)$

Examples. 1. All flags with ℓ on a given point.
2. All flags with ℓ in a given plane.

Either 24 or 16 relations depending on group.

Full group of A_3 give 16 relations: $\mathcal{A} \simeq 4\mathbb{C} \oplus 3\mathbb{C}^{2\times 2}$

Model: point-line flags in $Q^+(5, q)$ (using Klein correspondence)

Theorem (M, 2019+)

An EKR-set of chambers of $PG(3, q)$ has at most $(q^2 + q + 1)(q + 1)^2$ elements. For $q \geq 43$, only the above examples

EKR-sets of flags in classical situation

The classical set situation. Thin building of type A_n

- Let M be a finite set $|M| = n$.
- A flag is a chain

  $$T_1 \subset T_2 \subset \cdots \subset T_s$$

  of subsets o f M, its type is $I = \{|T_1|, \ldots, |T_s|\}$.
- Natural concept of general position.
- Problem: Find largest EKR-sets of flags of type I.

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- Problem: Find largest EKR-sets of flags of type I.
- Remark: reduces to classical case when $\max I \leq \frac{n}{2}$ or $\min I \geq \frac{n}{2}$.

Flags of type $\{1, n-2\}$

- Flags of type $\{1, n-2\}$ of the set $\{1, \ldots, n\}$, $n \geq 5$.
- Example $X(n, i)$, $1 \leq i \leq n-2$:
  All flags $\{A, B\}$ with $A = \{a\}$ and $|B| = n-2$ where
  - $a \leq i$ and $\{1, \ldots, a\} \subseteq B$, or
  - $\{1, \ldots, i\} \subseteq B \subseteq \{1, \ldots, n-1\}$
- This is maximal for $i = n-4$ and $i = n-5$ and comaximal for $i = n-3$.

Theorem (2019++)

Largest EKR-sets have size $|X(n, n-4)| = \binom{n}{3} + 2$.

Thank you very much for your attention!