DS or not DS?

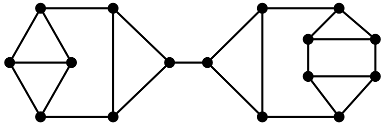
An application of Hoffman graphs for spectral characterizations of graphs

Aida Abiad

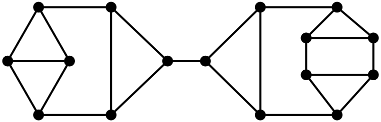
Maastricht University
The Netherlands

AEGT, 10 August 2017

Graph, adjacency matrix and spectrum



Graph, adjacency matrix and spectrum



$$A = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
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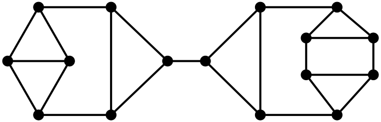
\end{bmatrix}$$

Graph, adjacency matrix and spectrum



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0&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0
\end{bmatrix}$$

spectrum (eigenvalues):

$\{\textbf{-2.0391}, -2^2, -1.64, -1.62, -1^2, -0.62, -0.11, 0, 0.62, 1.62, 1.67, 2.22, 2.89, 3\}$

A (finite simple) graph G on n vertices

$$\Downarrow$$

The spectrum $\lambda_1 \geq \cdots \geq \lambda_n$ of the adjacency matrix A of G

A (finite simple) graph G on n vertices

$\Uparrow$???

The spectrum $\lambda_1 \geq \cdots \geq \lambda_n$ of the adjacency matrix A of G

DS graphs

A graph G is said to be *determined by its spectrum (DS)* if every graph with the same spectrum as G is isomorphic to G.

DS or not DS?

Conjecture [van Dam and Haemers, 2003]
Almost all graphs are determined by their spectrum (DS).

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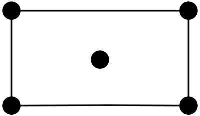
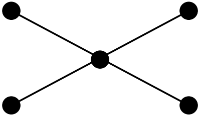
Cospectral graphs





$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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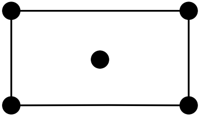

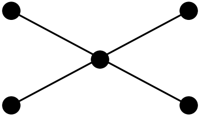

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

spectrum (eigenvalues): $\{-2, 0^3, 2\}$

Part I

non-DS: coclique extension

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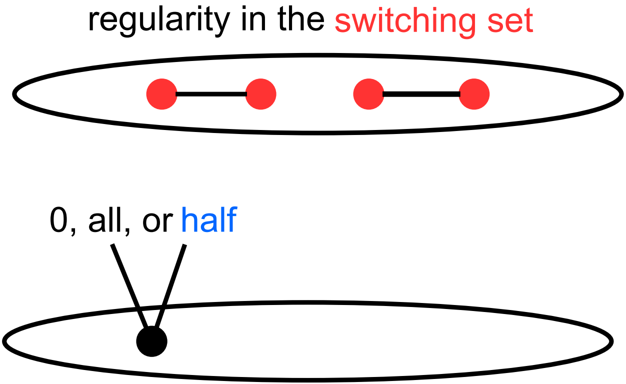
Joint work with A. Brouwer and W. Haemers

Coauthors

Godsil-McKay switching



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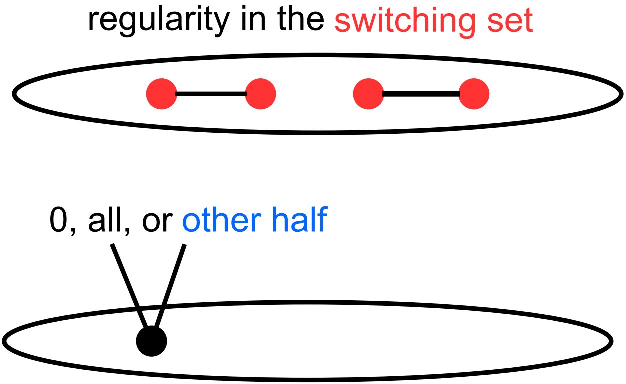

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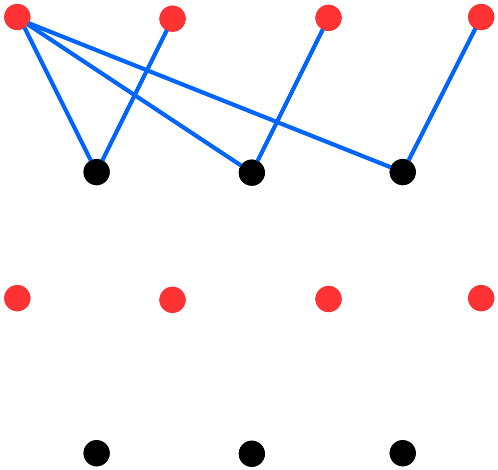

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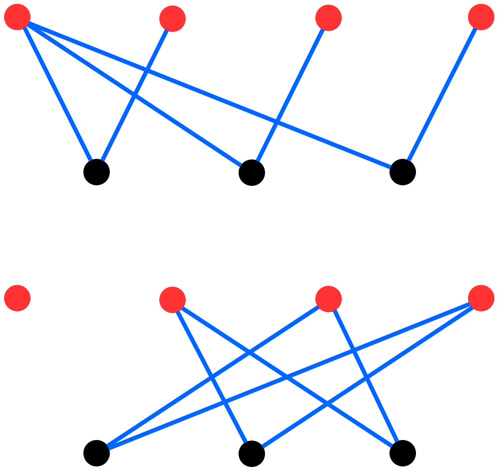

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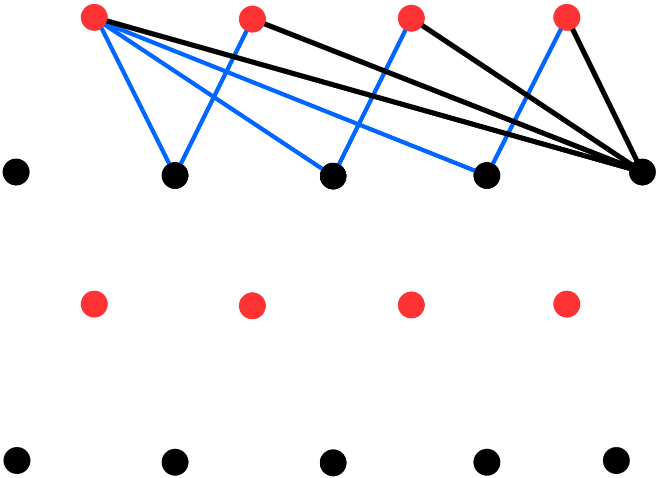

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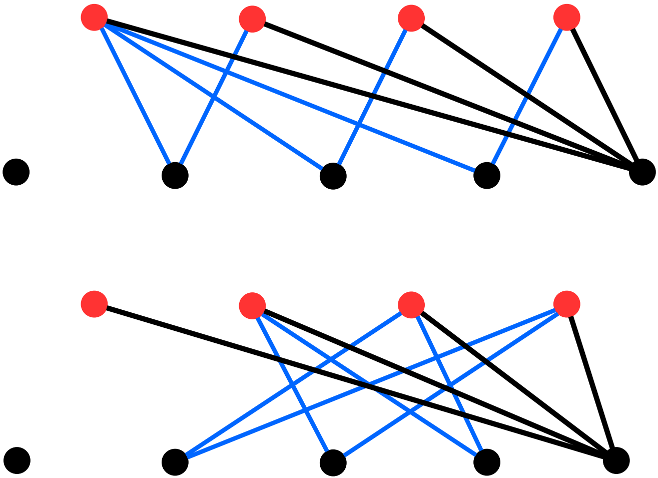

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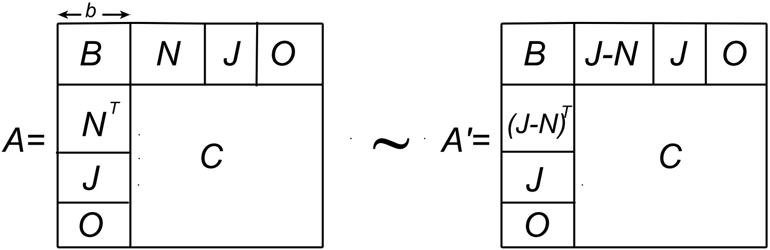

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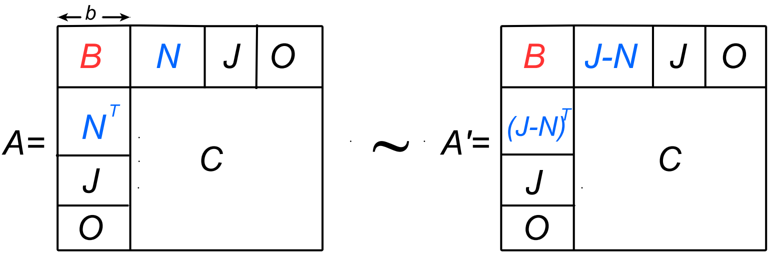

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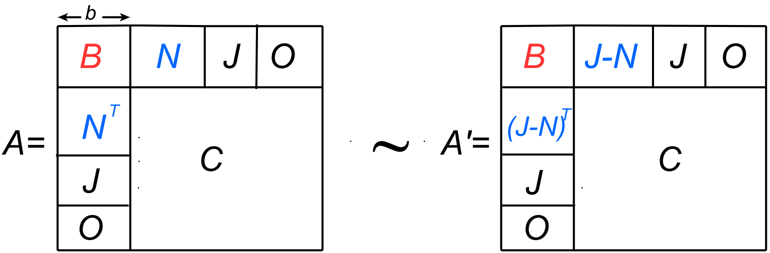
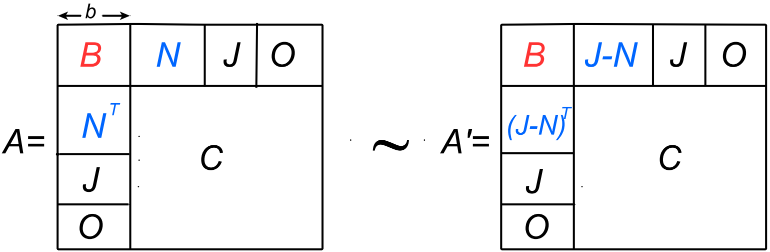


B: regular

$$N^\top J = (J - N)^\top J = \frac{b}{2} J$$

Godsil-McKay switching



B: regular

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A and A' cospectral

Goal

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Find conditions for isomorphism and non-isomorphism after switching.

Sufficient condition for isomorphism after switching

$$A = \begin{bmatrix} B & M \\ M^\top & C \end{bmatrix} \qquad A' = \begin{bmatrix} B & M' \\ M'^\top & C \end{bmatrix}$$

Sufficient condition for isomorphism after switching

$$A = \begin{bmatrix} B & M \\ M^\top & C \end{bmatrix} \qquad A' = \begin{bmatrix} B & M' \\ M'^\top & C \end{bmatrix}$$

Lemma (Abiad, Brouwer, Haemers, 2015)

If there exist permutation matrices P and Q such that $PBP^\top = B$, $PMQ^\top = M'$ and $QCQ^\top = C$, then G and G' are isomorphic.

Isomorphism fixing the switching set?

$$
\left[\begin{array}{cccc|cccccc}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\sim
\left[\begin{array}{cccc|cccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
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0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
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1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
$$

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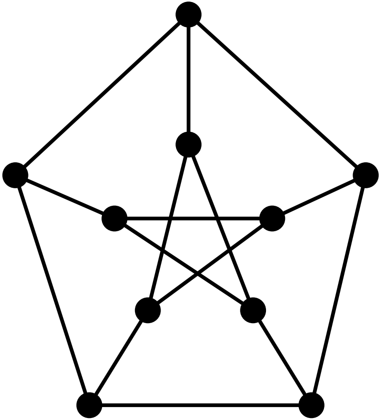

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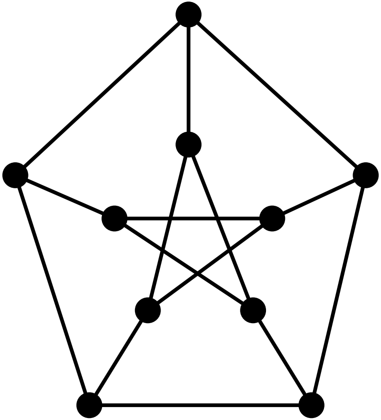

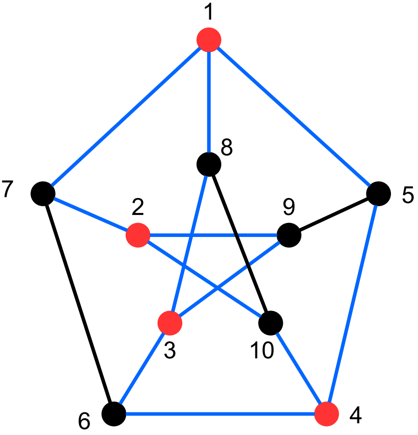

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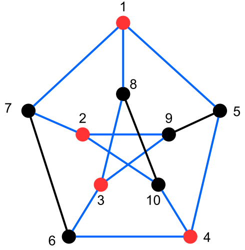

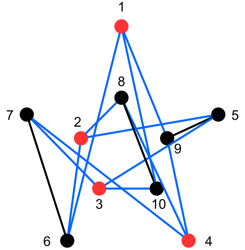

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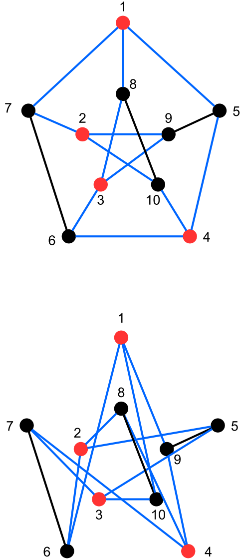

1 : 1
2 : 4
3 : 2
4 : 3
5 : 10
6 : 5
7 : 9
8 : 6
9 : 8
10 : 7

Isomorphism fixing the switching set?

No. No isomorphism fixes the switching set!

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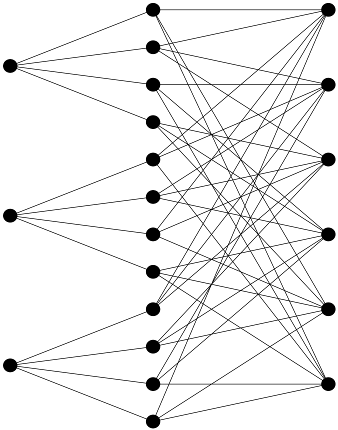

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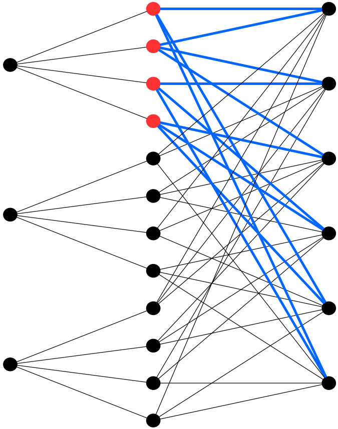

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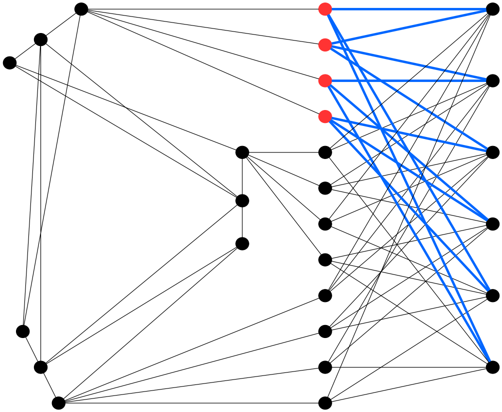
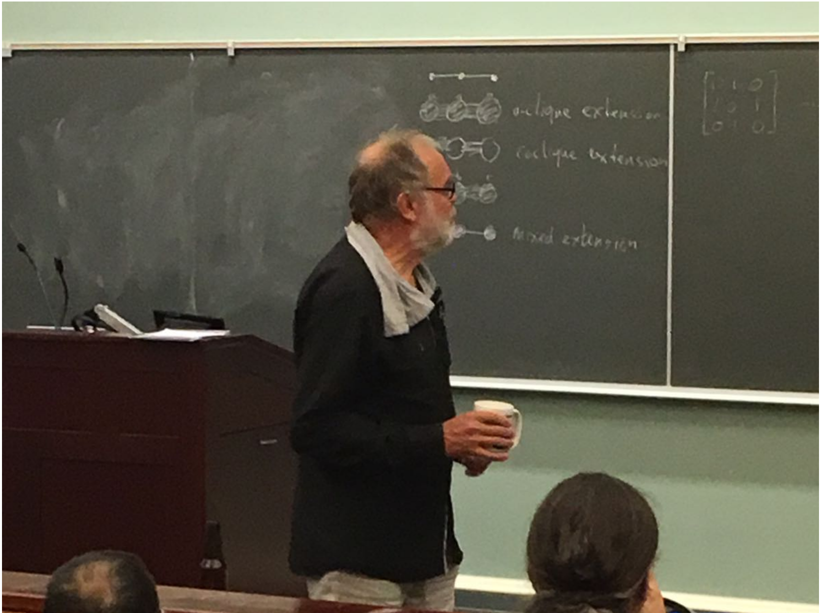


Graph products



Graph product: q-coclique extension

Definition

The q-**coclique extension** of Γ is the graph with adjacency matrix $A \otimes J$, where A is the adjacency matrix of Γ, J is a square all-ones matrix and $\otimes$ stands for the Kronecker product.

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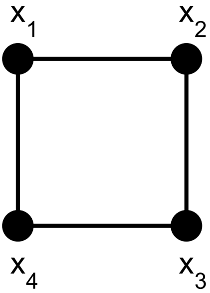
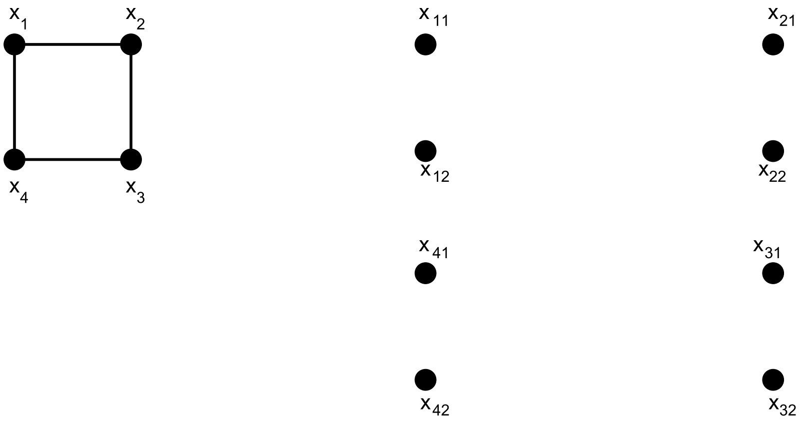
$$\begin{bmatrix} A & \cdots & A \\ \vdots & \ddots & \vdots \\ A & \cdots & A \end{bmatrix}$$

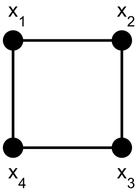
2-coclique extension of the grid

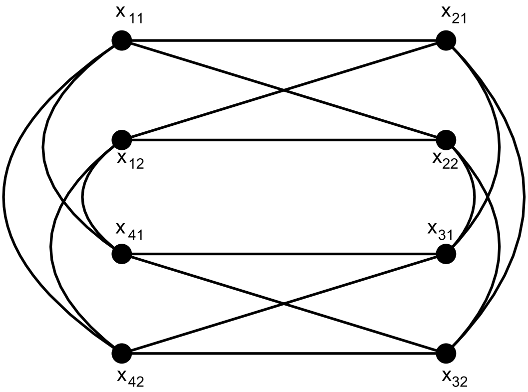


2-coclique extension of the grid

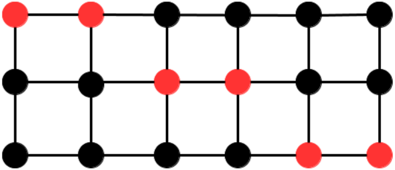


2-coclique extension of the grid





Regular example: 3×6 grid



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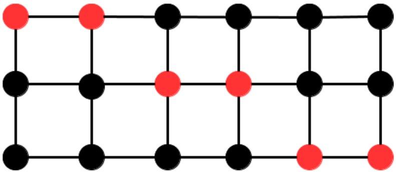

Does not generalize

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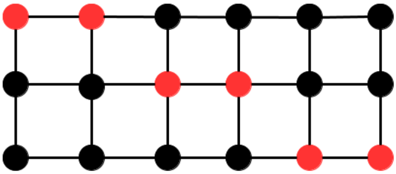

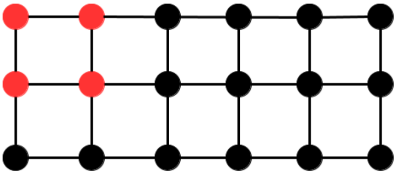

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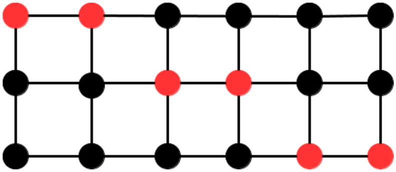

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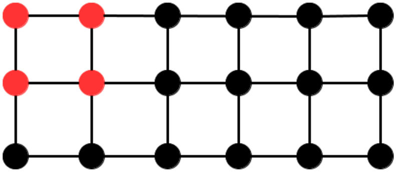

Does generalize

2-coclique extension of a grid

Corollary (Abiad, Brouwer, Haemers, 2015)

The q-coclique extension of a grid is not DS.

2-coclique extension of a grid

Corollary (Abiad, Brouwer, Haemers, 2015)

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q-clique extension of a grid???

Our results

- Straightforward sufficient condition for being isomorphic after switching

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- For some graph products, sufficient conditions for being non-isomorphic after switching

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- Straightforward sufficient condition for being isomorphic after switching
- For some graph products, sufficient conditions for being non-isomorphic after switching
- Application to some graph families to show that they are not determined by their spectrum

Open problems

- Find new sufficient conditions for (non)-isomorphism after Godsil-McKay switching.

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- Study other graph products.
- Find conditions for the generalized Godsil-McKay switching.

Reference

Godsil-McKay Switching and Isomorphism

Aida Abiad*, Andries E. Brouwer, Willem H. Haemers*
*Tilburg University, Tilburg, The Netherlands
(e-mails: A.AbiadMonge@uvt.nl, aeb@cwi.nl, haemers@uvt.nl)

Abstract

Godsil-McKay switching is an operation on graphs that doesn't change the spectrum of the adjacency matrix. Usually (but not always) the obtained graph is non-isomorphic with the original graph. We present a straightforward sufficient condition for being isomorphic after switching, and give examples which show that this condition is not necessary. For some graph products we obtain sufficient conditions for being non-isomorphic after switching. As an example we find that the tensor product of the $\ell \times m$ grid ($\ell > m \geq 2$) and a graph with at least one vertex of degree two is not determined by its adjacency spectrum.

Keywords: Godsil-McKay switching; Spectral characterization; Cospectral graphs; Graph isomorphism; Graph products.

Part II

DS: clique extension

Part II

DS: clique extension

Joint work with J. Koolen and Q. Yang

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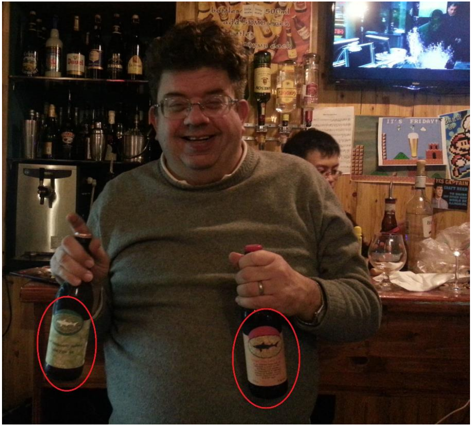

Motivation

Problem [Bannai, early 1980's]

Classify distance-regular graphs with large diameter.

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Classify distance-regular graphs with large diameter.

One of the steps towards a solution of Bannai's problem is to characterize the known DRGs by their intersection array.

Motivation

Theorem (Gavrilyuk and Koolen, 2017+)

The local subgraph of a distance-regular graph with the same intersection numbers as $J_q(2D, D)$ has the **same spectrum as the q-clique extension of a certain square grid**.

Motivation

Theorem (Metsch, 1995)

The Grassman graph $J_q(n, D)$, $D > 2$ is determined by its intersection numbers with the following possible exceptions:

- $n = 2D$, $n = 2D \pm 1$,
- $n = 2D \pm 2$ *if* $q \in \{2, 3\}$,
- $n = 2D \pm 3$ *if* $q = 2$.

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- $n = 2D \pm 2$ *if* $q \in \{2, 3\}$,
- $n = 2D \pm 3$ *if* $q = 2$.

If we prove that the 2-clique extension of a square grid is DS, that could be used to show that certain Grassmann graphs are unique as distance-regular graphs.

Motivation

First application of Hoffman graphs for spectral characterizations.

Motivation

First application of Hoffman graphs for spectral characterizations.

⇩

New tool for spectral characterizations

Motivation

The $(t+1) \times (t+1)$-grid is DS if $t \neq 3$.

Goal

Goal

Show that the 2-clique extension of a square grid is DS.

Graph product: q-clique extension

Definition

The q-**clique extension** of Γ is the graph $\widetilde{\Gamma}$ obtained from Γ by replacing each vertex $x \in V(\Gamma)$ by a **clique** $\widetilde{X}$ with q vertices, such that $\tilde{x} \sim \tilde{y}$ (for $\tilde{x} \in \widetilde{X},\ \tilde{y} \in \widetilde{Y},\ \widetilde{X} \neq \widetilde{Y}$) in $\widetilde{\Gamma}$ if and only if $x \sim y$ in Γ.

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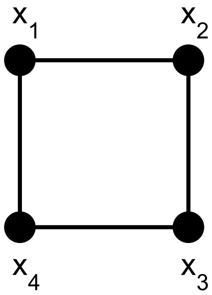
$$q = 2$$

2-clique extension of a $(t+1) \times (t+1)$ grid



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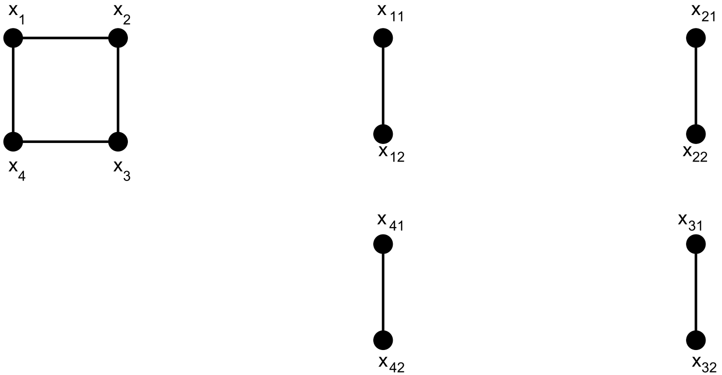

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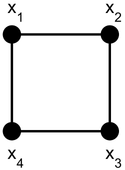

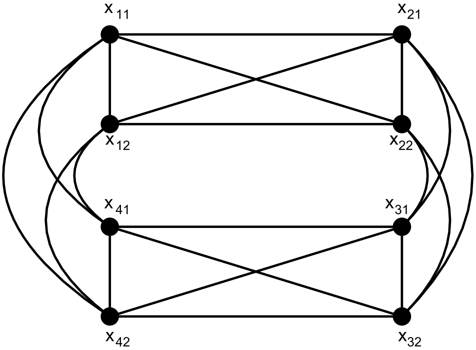

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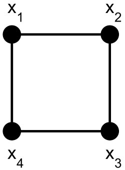

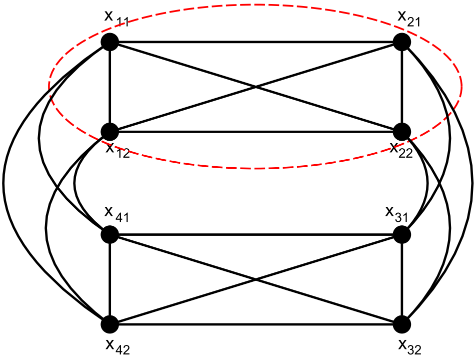

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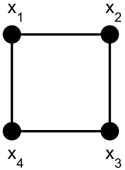

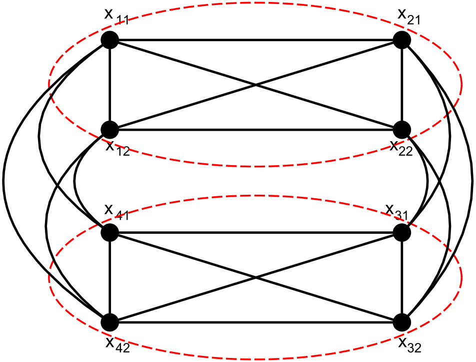

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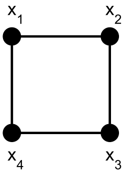

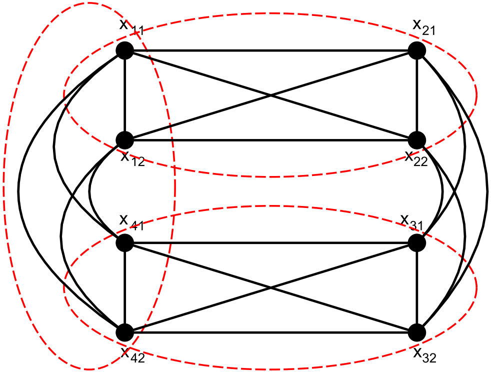

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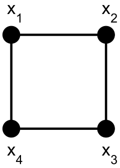

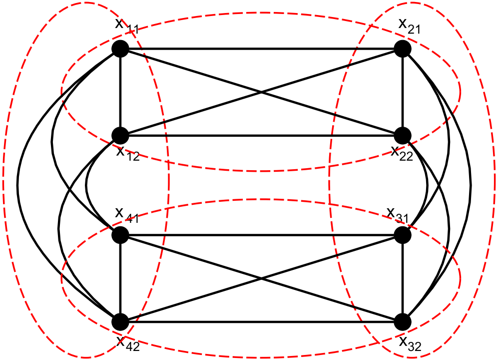

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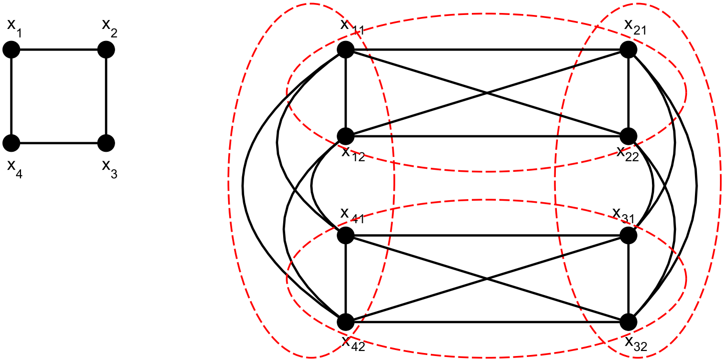

Regular graph with valency $k = 4t + 1$ and spectrum

$$\left\{(4t+1)^1,\ (2t-1)^{2t},\ (-1)^{(t+1)^2},\ (-3)^{t^2}\right\}.$$

Key observation

Regular graph with 4 distinct eigenvalues

Key observation

Regular graph with 4 distinct eigenvalues

Walk-regular graph

Bounded smallest eigenvalue

JOURNAL OF ALGEBRA 43, 305–327 (1976)

Line Graphs, Root Systems, and Elliptic Geometry

P. J. CAMERON*

Bedford College, London, England

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J. J. SEIDEL

Technological University, Eindhoven, Netherlands

AND

E. E. SHULT

Bounded smallest eigenvalue

Theorem (Cameron, Goethals, Seidel, Shult, 1976)

A graph G has smallest adjacency eigenvalue -2 if and only if G is a generalized line graph, or G belongs to a finite set of exceptional cases ($n \leq 36$).

Seidel tree


MOGE DEZE BOOM VOORTBLOEIEN
EN EEN DANKTEKEN ZIJN VOOR DE
VERDIENSTEN VAN
PROF. DR. J.J. SEIDEL
19 augustus 1919 8 mei 2001
DISPUUT JUPITER

Bounded smallest eigenvalue

ELSEVIER

Linear Algebra and its Applications

Volume 16, Issue 2, 1977, Pages 153-165

On graphs whose least eigenvalue exceeds $-1-\sqrt{2}$ ☆

A.J. Hoffman†

⊞ Show more

https://doi.org/10.1016/0024-3795(77)90027-1

Abstract

Let G be a graph, $A(G)$ its adjacency matrix. We prove that, if the least eigenvalue of $A(G)$ exceeds $-1-\sqrt{2}$ and every vertex of G has large valence, then the least eigenvalue is at least -2 and G is a generalized line graph.

Bounded smallest eigenvalue

Theorem (Hoffman, 1977)

Let $-1-\sqrt{2} < \lambda \leq -2$ be a real number. Then there exist an integer $f(\lambda)$ such that if G is a graph with smallest eigenvalue at least λ and minimum valency at least $f(\lambda)$, then G is a generalized line graph.

Bounded smallest eigenvalue

Bounded smallest eigenvalue

- In 1977, Hoffman gave a characterization of graphs with smallest eigenvalue at least -2.

Bounded smallest eigenvalue

- In 1977, Hoffman gave a characterization of graphs with smallest eigenvalue at least -2.
- **In 2017, Koolen, Yang and Yang have obtained a result for graphs with smallest eigenvalue at least -3.**

Bounded smallest eigenvalue

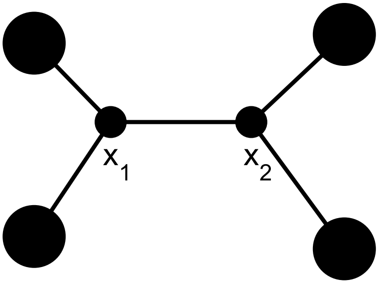
In order to bound the smallest eigenvalue, you need to obtain **some structure in the graph**.

Hoffman graphs

$\mathfrak{h}_1$

Hoffman graphs

$\mathfrak{h}_1$



Hoffman graphs

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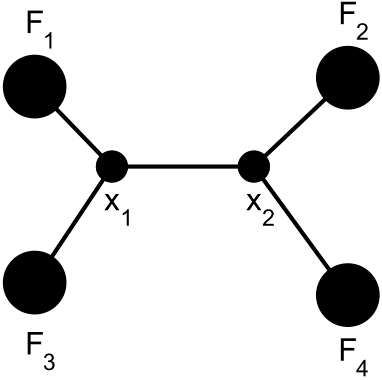

Hoffman graphs

$\mathfrak{h}_2$

Hoffman graphs

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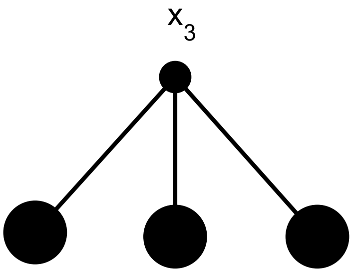

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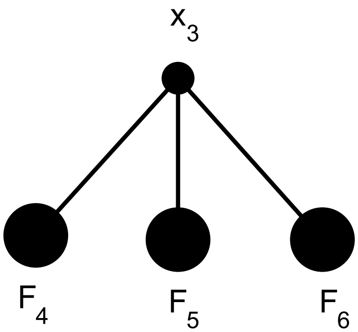

Hoffman graphs

Definition (Woo and Neumaier, 1995)

A *Hoffman graph* $\mathfrak{h}$ is a pair (H, μ) of a graph $H = (V, E)$ and a labeling map $\mu : V \to \{F, x\}$ satisfying the following conditions:

1. every fat vertex F is adjacent to at least one slim vertex x,
2. fat vertices F are pairwise non-adjacent.

Hoffman graphs

Hoffman graphs give a good **way to construct graphs with unbounded number of vertices such that the smallest eigenvalue is at least a fixed number**.

t-fat Hoffman graph

Definition

If every slim vertex has at least t fat neighbors, we call $\mathfrak{h}$ *t-fat*.

t-fat Hoffman graph

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$$t = 2$$

Slim graph of $\mathfrak{h}$

Definition

The *slim graph* of a Hoffman graph $\mathfrak{h}$ is the subgraph induced on the slim vertices of $\mathfrak{h}$.

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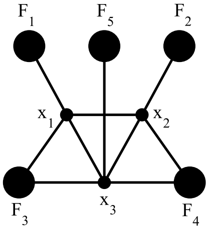
Example

$\mathfrak{h}$



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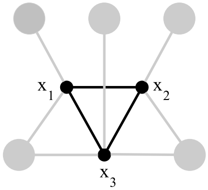
slim graph of $\mathfrak{h}$



Quasi-cliques of $\mathfrak{h}$

Definition

A *quasi-clique* $Q_{\mathfrak{h}}(F)$ is a subgraph induced by the neighbors of a fat vertex F of $\mathfrak{h}$.

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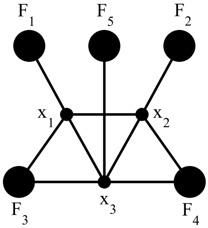

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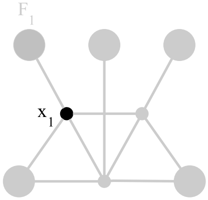
quasi-clique $Q_{\mathfrak{h}}(F_1)$



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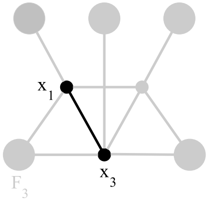
quasi-clique $Q_{\mathfrak{h}}(F_3)$



Special matrix

Let $\mathfrak{h}$ be a Hoffman graph.

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For $\mathfrak{h}$, let A be the adjacency matrix of H:

$$A := \begin{array}{c} \\ \end{array} \begin{array}{cc} \text{slim} & \text{fat} \\ \end{array}$$

$$A := \begin{pmatrix} A_s & C \\ C^\top & O \end{pmatrix}$$

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The *special matrix* is $S(\mathfrak{h}) := A_s - CC^\top$.
The eigenvalues of $\mathfrak{h}$ are the eigenvalues of $S(\mathfrak{h})$.

Special matrix

Let $x, y \in V_s(\mathfrak{h})$.

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$$S(\mathfrak{h})_{(x,y)} = \begin{cases} -|N^f_{\mathfrak{h}}(x)| & \text{if } x = y \\ 1 - |N^f_{\mathfrak{h}}(x,y)| & \text{if } x \sim y \\ 1 - |N^f_{\mathfrak{h}}(x,y)| & \text{if } x \not\sim y \end{cases}$$

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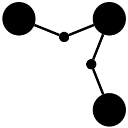
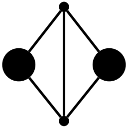


$$S := \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$



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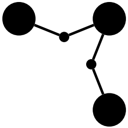

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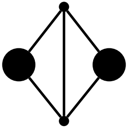

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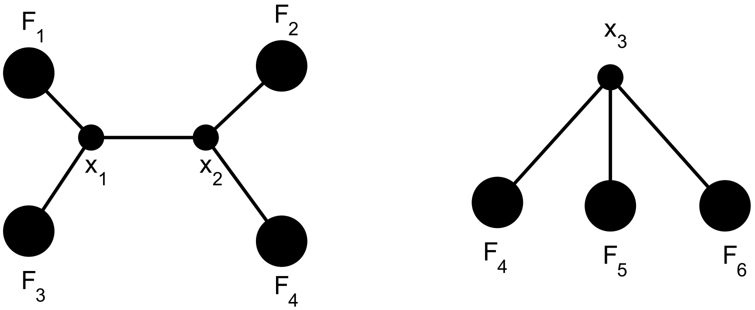
A Hoffman graph $\mathfrak{h}$ is not determined by $S(\mathfrak{h})$.

Direct sum: $\mathfrak{h}_1 \oplus \mathfrak{h}_2$



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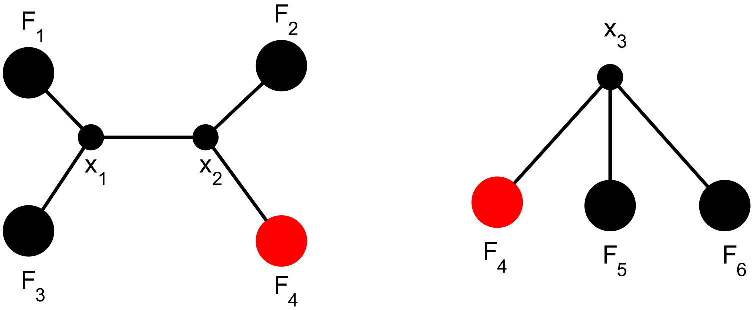

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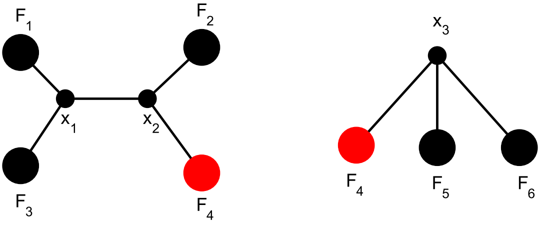

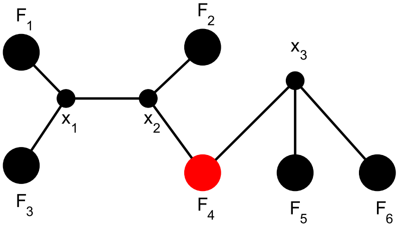

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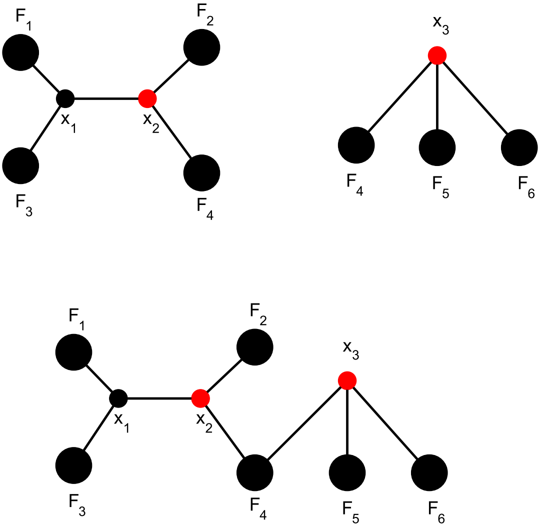

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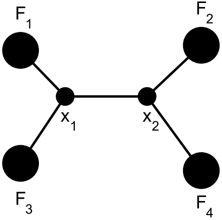
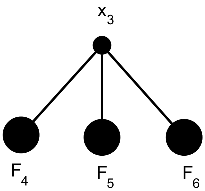
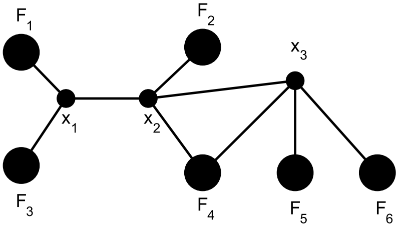
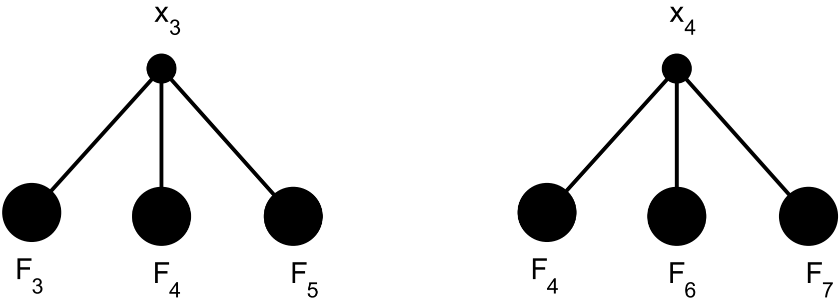






Direct sum: $\mathfrak{h}_3 \oplus \mathfrak{h}_4$



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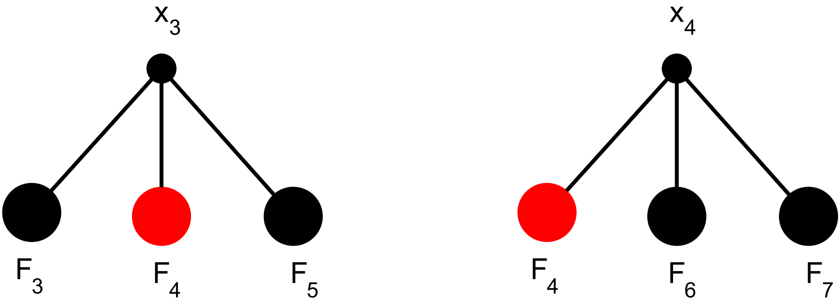

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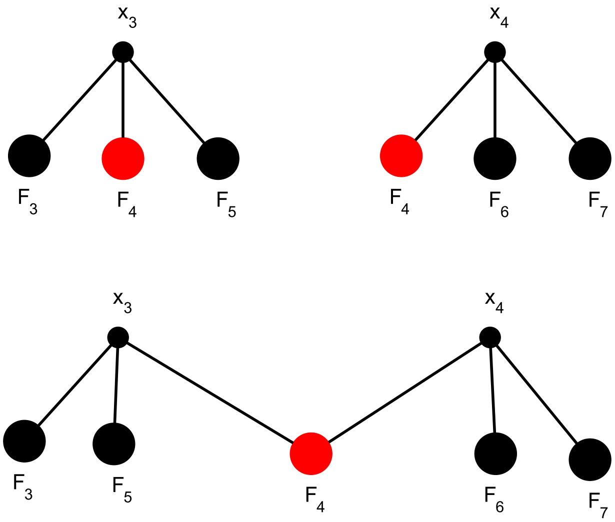

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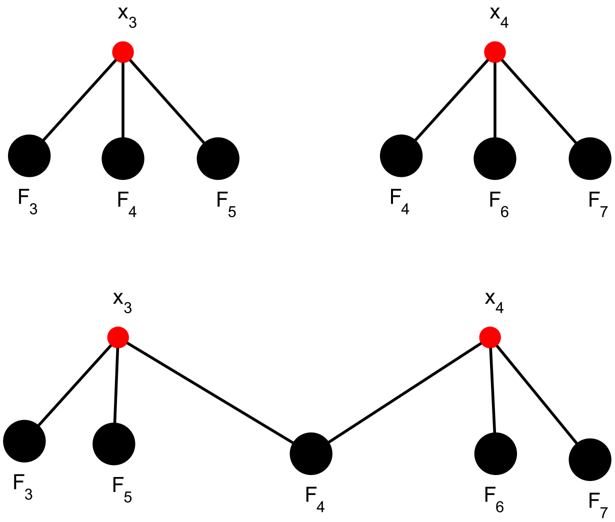

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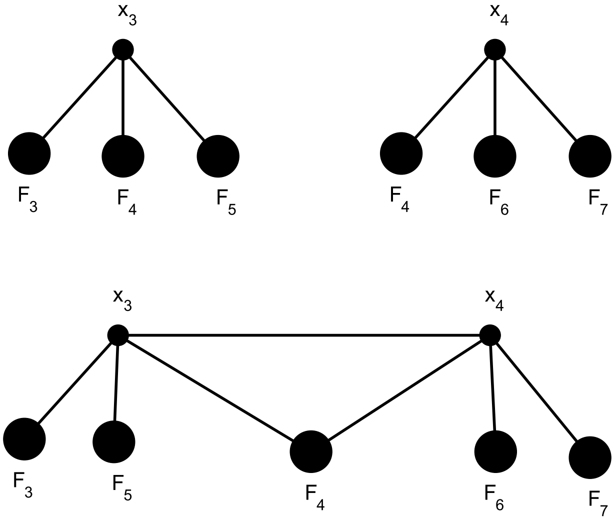

line Hoffman graph

Definition

Let $\mathcal{F}$ be a family of Hoffman graphs. A graph is called *$\mathcal{F}$-line graph* if it is an induced subgraph of the slim subgraph of $\oplus_{i=1}^{t}\mathfrak{F}_i$, where $\mathfrak{F}_i \in \mathcal{F}$.

line Hoffman graph

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Example

If $\mathfrak{h}$ is a 2-fat {, , }-line Hoffman graph

line Hoffman graph

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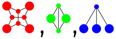
If $\mathfrak{h}$ is a 2-fat {  }-line Hoffman graph

$$\Downarrow$$

$\mathfrak{h}$ is a 2-fat Hoffman graph such that $\mathfrak{h} = \bigoplus_{i=1}^{s} \mathfrak{h}_i$, where $\mathfrak{h}_i$ is isomorphic to one of the Hoffman graphs in the set

$\mathcal{F} = \{$  $\}$ for $i = 1, \ldots, s$.

Hoffmania

Hoffmania

Longwood gardens in Pennsylvania

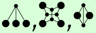
The starting point

Theorem (Koolen, Yang and Yang, 2017)

Let G be a graph with the same spectrum as the 2-clique extension of the $(t+1) \times (t+1)$-grid. Then, G is the slim graph of a 2-fat {, , }*-line Hoffman graph when $t \gg 0$.*

Our results

Theorem (Abiad, Koolen and Yang, 2017)

Let G be a graph with the same spectrum as the 2-clique extension of the $(t+1) \times (t+1)$-grid. If G is the slim graph of a 2-fat {}*-line Hoffman graph, then G is the 2-clique extension of the $(t+1) \times (t+1)$-grid when $t > 4$.*

Our results

Theorem (Abiad, Koolen and Yang, 2017)

The 2-clique extension of the $(t+1) \times (t+1)$-grid is characterized by its spectrum if $t \gg 0$.

Proof idea

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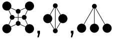
- G is a regular graph with valency $k = 4t + 1$ and spectrum

$$\left\{(4t+1)^1,\ (2t-1)^{2t},\ (-1)^{(t+1)^2},\ \mathbf{(-3)^{t^2}}\right\}$$

- G is the slim graph of a 2-fat {  }-line Hoffman graph (Koolen, Yang, Yang, 2017)

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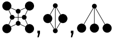
- G is the slim graph of a 2-fat {  }-line Hoffman graph (Koolen, Yang, Yang, 2017)

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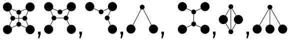
2. $\mathfrak{h} = \bigoplus_{i=1}^{s} \mathfrak{h}_i$, where $\mathfrak{h}_i$ is isomorphic to one of the Hoffman graphs in the set $\mathcal{F} = \{$  $\}$ for $i = 1, \ldots, s$.

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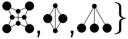
- G is the slim graph of a 2-fat {  }-line Hoffman graph (Koolen, Yang, Yang, 2017)

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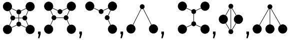
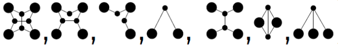
2. $\mathfrak{h} = \bigoplus_{i=1}^{s} \mathfrak{h}_i$, where $\mathfrak{h}_i$ is isomorphic to one of the Hoffman graphs in the set $\mathcal{F} = \{$  $\}$ for $i = 1, \ldots, s$.

$\mathcal{F} = \{$  $\}$

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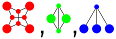
- G is the slim graph of a 2-fat {  }-line Hoffman graph (Koolen, Yang, Yang, 2017)

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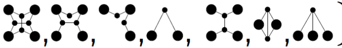
$\mathcal{F} = \{$  $\}$

Proof idea

Observation

- The Hoffman graph  has the same slim graph as .
- The Hoffman graph  has the same slim graph as , where  $=$  $\bigoplus$ .

$\mathcal{F} = \{$  $\}$

Proof idea

Forbidding  and 

Proposition

Any two disjoint vertices in G have at most $2t+2$ common neighbors.

$\mathcal{F} = \{$  $\}$

Proof idea

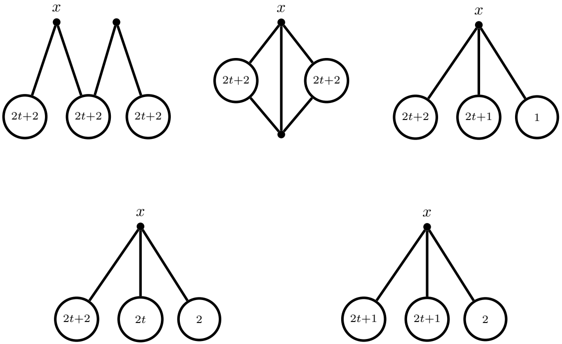
Determining the order of quasi-cliques

Proposition

Let q be the order of a quasi-clique $Q_{\mathfrak{h}}(F)$. Then $q \leq 2t + 2$ when $t > 1$.

Proof idea

Analysing the last candidates



Proof idea

Finishing the proof

- Counting in the local graphs

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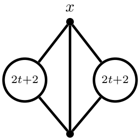

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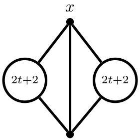

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Proof idea

Finishing the proof

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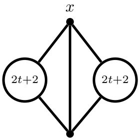


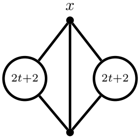
$\mathcal{F} = \{$  $\}$

- Equivalence relation on the vertex set

Proof idea

Finishing the proof

- Counting in the local graphs



$\mathcal{F} = \{$  $\}$

- Equivalence relation on the vertex set

⇩

G is the 2-clique extension of the $(t+1) \times (t+1)$-grid when $t > 4$

□

Open problems

- 2-clique extension of a non-square grid has 5 distinct eigenvalues $\implies$ same approach cannot be used

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Open problems

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- Study $q > 2$
- How large is $t \gg 0$?
- Find other applications of Hoffman graphs

Reference

An application of Hoffman graphs for spectral characterizations of graphs

Qianqian Yang[†], Aida Abiad[‡], Jack H. Koolen[†*]

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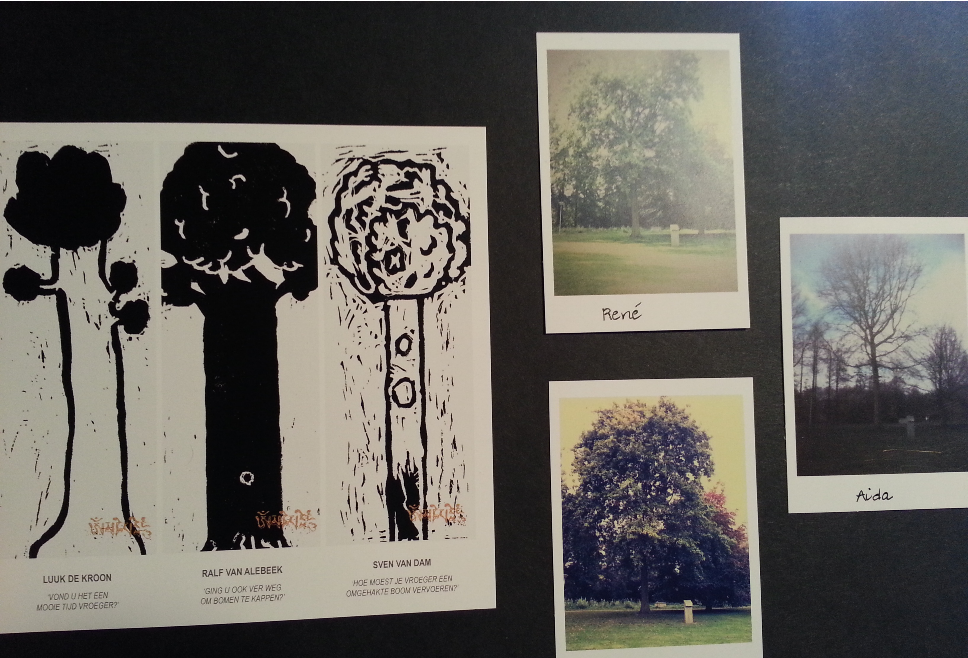
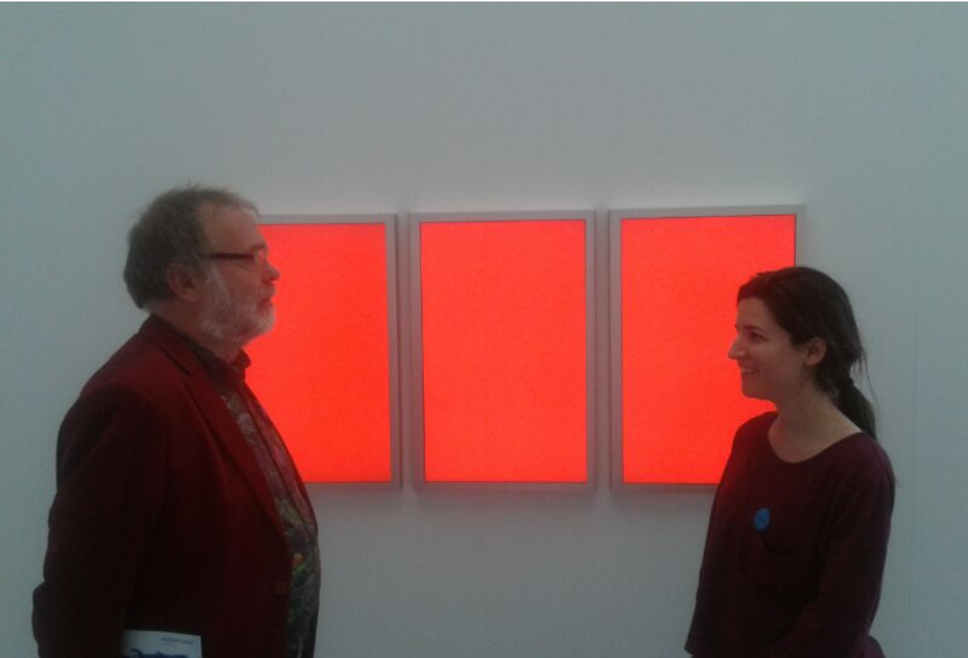
Haemers's tree



René

Edwin

Aida



DS or not DS?

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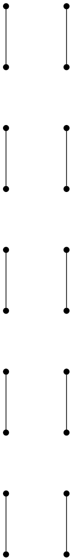
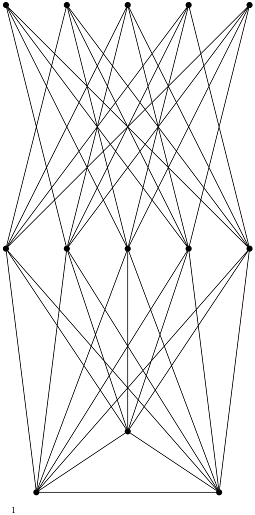

Congratulations Andrew, Felix and Willem!

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Thank you for your attention.