The Graphs $CD(k, q)$ and Their Relatives

Jason Williford
University of Wyoming

August 9th, 2017

Turán Type Problems

Definition

Let v be a positive integer and H a graph. We define $ex(v, H)$ to be the largest number of edges in a graph with v vertices which contains no copy of H as a subgraph.

For H of chromatic number 3 or greater, the asymptotic value is known.

Theorem

(Erdős, Stone, Simonovits)
$ex(v, H) \sim \left(1 - \frac{1}{\chi - 1}\right) \frac{v^2}{2}$, *where* $\chi > 2$ *is the chromatic number of* H.

Much less is known in the case where H is bipartite.

Even Cycles

Theorem (Bondy, Simonovits '74)

$ex(v, C_{2h}) \leq 90hv^{1+\frac{1}{h}}$

Theorem (Verstraëte 2000)

$ex(v, C_{2h}) \leq 8(h-1)v^{1+\frac{1}{h}}$

Theorem (Pikhurko 2012)

$ex(v, C_{2h}) \leq (h-1)v^{1+\frac{1}{h}} + O(v).$

Theorem (Bukh, Jiang 2017)

$ex(v, C_{2h}) \leq 80\sqrt{h}log(h)v^{1+\frac{1}{h}} + O(v)$

Definition

A generalized n-gon is a biregular bipartite graph of girth $2n$ and diameter n.

The only n for which a generalized n-gon that is also a regular graph exists is $n = 2, 3, 4, 6$, due to a theorem of Feit and Higman.

A generalized 3-gon is the incidence graph of a projective plane, generalized 4-gon the incidence graph of a generalized quadrangle, and a generalized 6-gon is the incidence graph of a generalized hexagon.

Constructive Lower Bounds

The best lower bounds (up to a constant) come from graphs known as generalized polygons:

Theorem

$ex(v, C_{2h}) \geq \frac{1}{2^{1+1/h}} v^{1+1/h}$ for $l = 2, 3, 5$.

Note that the exponent is optimal.

Theorem

(Lubotzky, Phillips, Sarnak 1988)
$ex(v, C_{2h}) \geq c_h v^{1+\frac{2}{3h+3}}$

Theorem

(Lazebnik, Ustimenko, Woldar 1995)
$ex(v, C_{2h}) \geq c_h v^{1+\frac{2}{3h-3+\epsilon}}$, where $\epsilon = 0, 1$ depending on whether h is odd or even.

The series of graphs $D(k, q)$

Vertices: Two copies of F_q^k, one called "Points", the other "Lines".

We have $p \sim l$ if and only if the following hold:

$p_2 + l_2 = p_1 l_1$
$p_3 + l_3 = p_1 l_2$
$p_4 + l_4 = -p_2 l_1$
...
$p_i + l_i = -p_{i-1} l_1$ if $i \equiv 0, 1 \bmod 4$
$p_i + l_i = p_1 l_{i-1}$ if $i \equiv 2, 3 \bmod 4$

The components of these graphs give the graphs $CD(k, q)$, which in turn yield $ex(n, C_{2h}) \geq c_h v^{1+\frac{2}{3h-3+\epsilon}}$, where $\epsilon = 0, 1$ depending on whether h is odd or even.

The Architects of $CD(k, q)$



From left to right: Felix Lazebnik, Vasyl Ustimenko, Andrew Woldar.

Best known lower bounds on extremal problems for even cycles $\neq 10, 14$.

Best known lower bounds on extremal problems for fixed girth $\neq 12$.

Valid for all characteristics.

Motivated by Ustimenko's embeddings of generalized polygons into respective Lie algebras.

Automorphism group is transitive on unordered 3-paths.

The end of the road?

Algebraically Defined Graphs of Lazebnik, Woldar

Let R be a ring and k a positive integer. Let $f_i : R^k \times R^k \to R^k$ be a sequence of functions, $i = 2, 3, \ldots, k-1$, such that $f_i(p, l)$ depends only on the first $i-1$ coordinates of p and l.

We define a bipartite graph $\Gamma(R, k, \{f_2, \ldots, f_k\})$ to have vertex set equal to the union of two copies P and L of R^k. We refer to elements of P as points, and elements of L as lines.
For $p \in P$ and $l \in L$ we have $p \sim l$ if and only if

$$p_i + l_i = f_i(p, l) \text{ for all } i \in \{2, 3, \ldots, k\}$$

Affine Parts of Generalized n-gons

Let F_q be a finite field. The affine part of a classical projective plane is given as an ADG by:

$$p_2 + l_2 = p_1 l_1$$

The affine part of a classical generalized quadrangle is given as an ADG by:

$$p_2 + l_2 = p_1 l_1$$
$$p_3 + l_3 = p_1 l_2$$

Affine Parts of Generalized n-gons

The affine part of a classical generalized hexagon is given as an ADG by:

$$p_2 + l_2 = p_1 l_1$$
$$p_3 + l_3 = p_1 l_2$$
$$p_4 + l_4 = p_1 l_3$$
$$p_5 + l_5 = p_3 l_2 - p_2 l_3$$

The automorphism group of each corresponding graph is transitive on unordered 3-paths.

Wenger Graphs

A series of graphs based on a graph of Wenger:

$$p_2 + l_2 = p_1 l_1$$
$$p_3 + l_3 = p_1 l_2$$
$$p_4 + l_4 = p_1 l_3$$
$$\ldots$$
$$p_k + l_k = p_1 l_{k-1}$$

Similarly the automorphism group of each corresponding graph is transitive on 3-paths. For $k = 2, 3$ these graphs have girth $6, 8$, and are isomorphic to affine parts of projective planes and generalized quadrangles. For $k = 5$, this graph has no 10 cycles, but has 8 cycles.

The graph $\Gamma(R, k, \{f_2, \ldots, f_k\})$ is $|R|$-regular and has $2|R^k|$ vertices.

In particular, given a vertex p and an $x \in F$, there is a unique neighbor of p with first coordinate x. This can be found by recursively computing the coordinates of the neighbor from the equations $p_i + l_i = f_i(p, l)$.

Theorem

(Lazebnik, Woldar '01) Let $\Gamma_1 = \Gamma(R, k, \{f_2, \ldots, f_k\})$ *and* $\Gamma_2 = \Gamma(R, k-1, \{f_2, \ldots, f_{k-1}\})$. *There is a surjective, locally injective homomorphism from* Γ_1 *to* Γ_2 *given by puncturing the last coordinate of every vertex of* Γ_1. *In particular, the girth of* Γ_1 *is greater than or equal to the girth of* Γ_2.

The following ADG has $2q^2$ vertices, q^3 edges and girth 6:
$p_2 + l_2 = p_1 l_1$

Algebraically Defined Graphs

Theorem

(Lazebnik, Woldar '01) Let $\Gamma_1 = \Gamma(R, k, \{f_2, \dots, f_k\})$ and $\Gamma_2 = \Gamma(R, k-1, \{f_2, \dots, f_{k-1}\})$. There is a surjective, locally injective homomorphism from Γ_1 to Γ_2 given by puncturing the last coordinate of every vertex of Γ_1. In particular, the girth of Γ_1 is greater than or equal to the girth of Γ_2.

The following ADG has $2q^3$ vertices, q^4 edges and girth 8:

$p_2 + l_2 = p_1 l_1$

$p_3 + l_3 = p_1 l_2$

Algebraically Defined Graphs

Theorem

(Lazebnik, Woldar '01) Let $\Gamma_1 = \Gamma(R, k, \{f_2, \ldots, f_k\})$ *and* $\Gamma_2 = \Gamma(R, k, \{f_2, \ldots, f_{k-1}\})$. *There is a surjective, locally injective homomorphism from* Γ_1 *to* Γ_2 *given by puncturing the last coordinate of every vertex of* Γ_1. *In particular, the girth of* Γ_1 *is greater than or equal to the girth of* Γ_2.

The following ADG has $2q^4$ vertices, q^5 edges and girth 8:

$p_2 + l_2 = p_1 l_1$
$p_3 + l_3 = p_1 l_2$
$p_4 + l_4 = p_1 l_3$

Theorem

(Lazebnik, Woldar '01) Let $\Gamma_1 = \Gamma(R, k, \{f_2, \ldots, f_k\})$ *and* $\Gamma_2 = \Gamma(R, k, \{f_2, \ldots, f_{k-1}\})$*. There is a surjective, locally injective homomorphism from* Γ_1 *to* Γ_2 *given by puncturing the last coordinate of every vertex of* Γ_1*. In particular, the girth of* Γ_1 *is greater than or equal to the girth of* Γ_2*.*

The following ADG has $2q^5$ vertices, q^6 edges and girth 12:

$p_2 + l_2 = p_1 l_1$
$p_3 + l_3 = p_1 l_2$
$p_4 + l_4 = p_1 l_3$
$p_5 + l_5 = p_3 l_2 - p_2 l_3$

Cycles and Gröbner Bases

Suppose there is a cycle of length $2k$ is in some ADG Γ. We can describe this cycle as a system of polynomial equations. If we show the associated variety is empty, then there is no such cycle.

For example, to show $p_2 + l_2 = p_1 l_1$ has no 4-cycles, we can solve:

$$p_2 + l_2 - p_1 l_1 = 0$$
$$p_2 + m_2 - p_1 m_1 = 0$$
$$q_2 + l_2 - q_1 l_1 = 0$$
$$q_2 + m_2 - q_1 m_1 = 0$$
$$1 - k(p_1 - q_1)(l_1 - m_1) = 0$$

ADG's

Pros: Contains all known examples, big family, lots of room for things to exist.

Con: Big family, unclear how we find the "good graphs"?

Woldar: Go back to the Lie algebraic connections.

Background on Lie Algebras

A Lie algebra $\mathcal{L}$ is a vector space V together with a product $[,] : V \times V \to V$ that satisfies:

1. $[,]$ is bilinear
2. $[x, x] = 0$ for all vectors x
3. (Jacobi identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Note that the first two axioms imply that $[x, y] = -[y, x]$.

If $[x, y] = 0$, we say x and y "commute".

Examples of Lie Algebras

Example 1: The cross product in $\mathbb{R}^3$.

Example 2: $M_n(\mathbb{F})$ with $[A, B] = AB - BA$.

Example 3: Given a vector space V, $\mathfrak{gl}(V)$ consists of all linear operators on V with Lie bracket $[S, T] = ST - TS$

Example 4: $\mathfrak{sl}(V)$ is the subalgebra of $\mathfrak{gl}(V)$ consisting of all elements with trace zero.

Example 5: An associative algebra with $[x, y] = xy - yx$.

Adjoint Representations

Given an element x of a Lie algebra $\mathcal{L}$, we can define the adjoint map $ad(x) : \mathcal{L} \to \mathcal{L}$ via $ad(x)(y) = [x, y]$.

Adjoints give a convenient way to represent repeated Lie products: $[x, [x, [x, y]]] = ad(x)^3(y)$

The map $ad(x)$ is a linear operator on V.

The adjoint map has the property:
$[ad(x), ad(y)](z) = ad([x, y])(z)$, where
$[ad(x), ad(y)] = ad(x) \circ ad(y) - ad(y) \circ ad(x)$.

The implies that the map $ad : \mathcal{L} \to \mathfrak{gl}(\mathcal{L})$ is a Lie algebra homomorphism. The kernel of this homomorphism is the center of $\mathcal{L}$.

Nilpotent elements

An element x of a Lie algebra is called nilpotent provided that there is an integer n such that $ad(x)^n = 0$.

If x is nilpotent and $\delta = ad(x)$, and the characteristic of the field is zero or sufficiently large, then the exponential map $exp(x) = \sum_{k=0}^{\infty} \frac{\delta^k}{k!}$ is well-defined, invertible, and is an automorphism of $\mathcal{L}$.

We have $exp(x)([y, z]) = [exp(x)(y), exp(x)(z)]$.

Example

The Lie algebra $\mathfrak{sl}(\mathbb{F}^3)$ is spanned by:

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \; e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \; f_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We have $[e_1, e_2] = e_3$, $[f_1, f_2] = -f_3$, $[h_1, h_2] = [e_1, f_2] = [e_2, f_1] = 0$, $[e_1, f_1] = h_1$, $[e_2, f_2] = h_2$, $[h_1, e_1] = 2e_1$, $[h_2, e_2] = 2e_2$, $[h_1, e_2] = -e_2$, $[h_2, e_1] = -e_2$. Also $[e_1, [e_1, e_2]] = ad(e_1)^2(e_2) = 0$, $ad(f_1)^2(f_2) = 0$.

A Family of Generalized Kac-Moody Algebras

Let C be a 2 by 2 generalized Cartan matrix, i.e. an integral matrix with $C_{11} = C_{22} = 2$ and $C_{12}, C_{21} < 0$. We let $\mathcal{F}(F)$ be the free Lie algebra generated by the variables h_1, h_2, e_1, e_2 over the field F. Let $\mathcal{L}$ to be the quotient of $\mathcal{F}$ by the relations:

1. $[h_i, h_j] = 0$
2. $[h_i, e_j] = \delta_{ij} e_j$
3. $ad(e_i)^{1-C_{ij}}(e_j) = 0$

If C is not positive definite, the Lie algebra will be infinite dimensional.

Cartan Matrices

The following are the positive definite 2×2 Cartan matrices:

$$M_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \; M_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \; M_3 = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

One can drop the condition that C is positive definite, however the resulting Lie algebra is infinite dimensional.

$$M_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \; M_2 = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix},$$

These Lie algebras, called Kac-Moody algebras, have many finite dimensional quotients.

Example

If we take $M_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, we obtain a Lie algebra spanned by $h_1, h_2, e_1, e_2, e_3 = [e_1, e_2], e_4 = [e_1, [e_1, e_2]]$.

The multiplication table for this algebra is:

	h_1	h_2	e_1	e_2	e_3	e_4
h_1	0	0	$2e_1$	$-2e_2$	0	$2e_4$
h_2	0	0	$-e_1$	$2e_2$	e_3	0
e_1	$-2e_1$	e_1	0	e_3	e_4	0
e_2	$2e_2$	$-2e_2$	$-e_3$	0	0	0
e_3	0	e_3	$-e_4$	0	0	0
e_4	$-2e_4$	0	0	0	0	0

The subalgebra $\mathcal{L}^+$

Let $\mathcal{L}^+$ be the subalgebra generated by e_1, e_2.

We define a word in $\mathcal{L}^+$ to be an expression involving the generators e_1, e_2 and the Lie bracket. The length of the word is the number of generators it contains.

We can define a basis $\{w_1, w_2, \dots\}$ of nonzero words algebra $\mathcal{L}^+$ such that the length of the words w_i is nondecreasing. To obtain a finite dimensional Lie algebra, we may quotient by all words w_i for $i \geq n$ for some fixed n. We will denote this by $\mathcal{L}_n$.

Let $\mathcal{L}_n$ be a finite dimensional quotient algebra of $\mathcal{F}$, satisfying the previous relations.

We let $\mathcal{L}_n^+$ be the subalgebra generated by e_1, e_2, and let $\mathcal{A}, \mathcal{B}$ be the ideals of $\mathcal{L}_n$ generated by e_1 and e_2, respectively.

Let P to be the set of vectors of $\mathcal{L}$ in the coset $-h_1 + \mathcal{A}$ and L be the set of vectors n the coset $-h_2 + \mathcal{B}$.

We define the bipartite graph $\Gamma(\mathcal{L}_n)$ to have bipartition P and L with $p \in P$, $l \in L$ adjacent if and only if $[p, l] = 0$.

Lie Graphs and Generalized Polygons

If one takes the following matrices:

$$M_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},\ M_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix},\ M_3 = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

the corresponding Lie graphs are isomorphic to the affine parts of the generalized triangles (projective planes), generalized quadrangles and generalized hexagons, for fields of large enough characteristic.

The affine part of a generalized quadrangle

If we take $M_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, we obtain a Lie algebra spanned by $h_1, h_2, e_1, e_2, e_3 = [e_1, e_2], e_4 = [e_1, [e_1, e_2]]$.

The multiplication table for this algebra is:

	h_1	h_2	e_1	e_2	e_3	e_4
h_1	0	0	e_1	0	e_3	$2e_4$
h_2	0	0	0	e_2	e_3	e_4
e_1	$-e_1$	0	0	e_3	e_4	0
e_2	0	$-e_2$	$-e_3$	0	0	0
e_3	$-e_3$	$-e_3$	$-e_4$	0	0	0
e_4	$-2e_4$	$-e_4$	0	0	0	0

The affine part of a generalized quadrangle

Points: $-h_1 + p_1 e_1 + p_2 e_3 + p_3 e_4$
Lines: $-h_2 + l_1 e_2 + l_2 e_3 + l_3 e_4$

We have p adjacent to l iff $[p, l] = 0$, which occurs when $(p_2 - l_2 + p_1 l_1)e_3 + (p_3 - 2l_3 + p_1 l_2)e_4 = 0$. This gives the equations:

$$p_2 - l_2 + p_1 l_1 = 0$$
$$p_3 - 2l_3 + p_1 l_2 = 0$$

After some changes of variables, we obtain the following ADG:

$$p_2 + l_2 = p_1 l_1$$
$$p_3 + l_3 = p_1 l_2$$

Points: $-h_1 + p_1 e_1 + p_2 e_3 + p_3 e_4$
Lines: $-h_2 + l_1 e_2 + l_2 e_3 + l_3 e_4$

The element e_1 is nilpotent in $\mathcal{L}_n$, in particular $ad(e_1) = \delta$ satisfies $\delta^3 = 0$. So the map $\alpha = 1 + \delta + \frac{\delta^2}{2}$ is an automorphism of $\mathcal{L}_n$.

We have
$\alpha(-h_1 + p_1 e_1 + p_2 e_3 + p_3 e_4) = -h_1 + (p_1 + 1)e_1 + p_2 e_3 + (p_2 + p_3)e_4$, so α preserves points. A similar calculations shows that lines are preserved as well.

Theorem (Terlep, W 2012)

Suppose there is no nonzero word w in the subalgebra $\mathcal{L}_n^+$ which satisfies $[h_1, w] = 0$ or $[h_2, w] = 0$. Then the corresponding Lie Graph are ADG's. Furthermore the automorphism group of this graph is transitive on unordered 3-paths for sufficiently large characteristic p and characteristic zero.

Suppose we take the following matrix and construct the associated Lie graphs:

$$M_4 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$p_2 - l_2 = p_1 l_1$$
$$2p_3 - l_3 = p_2 l_1$$
$$p_4 - 2l_4 = -p_1 l_2$$
$$2p_5 - 2l_5 = -p_1 l_3 + p_4 l_1$$
$$3p_6 - 2l_6 = -p_2 l_3 + p_3 l_2 + p_5 l_1$$
$$2p_7 - 3l_7 = -p_1 l_5 + p_2 l_4 - p_4 l_2$$
$$3p_8 - 3l_8 = -p_1 l_6 + p_3 l_4 - p_4 l_3 + p_7 l_1$$

Suppose we take the following matrix and construct the associated Lie graphs

$$M_4 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

Conjecture

For each $n, t \geq 1$ and sufficiently large prime p, $\Gamma(\mathcal{L}_n, p^t)$ is isomorphic to $CD(k, p^t)$ for an appropriate choice of k.

Now suppose we take the matrix $M_4 = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

We consider $\Gamma(\mathcal{L}_8)$, and obtain the equations:

$$p_2 + l_2 = p_1 l_1$$

$$p_3 + l_3 = p_1 l_2$$

$$p_4 + l_4 = p_1 l_3$$

$$p_5 + l_5 = p_1 l_4$$

$$p_6 + l_6 = p_2 l_3 - 2p_3 l_2 + p_4 l_1$$

$$p_7 + l_7 = p_1 l_6 + p_2 l_4 - 3p_4 l_2 + 2p_5 l_1$$

$$p_8 + l_8 = 2p_2 l_6 - 3p_6 l_2 + p_7 l_1$$

A Lie Graph With No Cycles of Length Fourteen

Theorem (Terlep, W 2012)

For sufficiently large primes p and all q which are powers of p,

$$ex(n, C_{14}) \geq \frac{1}{2^{9/8}} n^{1+\frac{1}{8}}, \text{ where } n = 2q^8.$$

We note that these graphs have girth 12. The lack of 14-cycles was shown by a computer using Groebner bases.

The previous bound was $ex(n, C_{14}) \geq \frac{1}{2^{10/9}} n^{1+\frac{1}{9}}$, achieved by $CD(12, q)$ and by a group theoretic construction of Ustimenko and Woldar.

Open Questions

- Computer free proof of C_{14} result? Other missing cycles in this or other families?
- Proof that first matrix gives $CD(k, q)$ for sufficiently large q relative to k?
- Direct use of Lie algebra in computation of cycle spectrum?
- More direct use of Lie algebra in computation of cycle spectrum?
- Classify ADG's that are transitive on ordered 3-paths?
- More direct use of Lie algebra in computation of cycle spectrum?

Thank You!