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Eigenvalues and eigenvectors of the perfect matching association scheme

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$\mathcal{M}_{2n}$ = set of all **perfect matchings** in the complete graph K_{2n}.

$|\mathcal{M}_{2n}| = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

S_{2n} has a natural substitution action on $\mathcal{M}_{2n}$:

$$\pi \cdot (\{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}) = \{\{\pi(i_1), \pi(j_1)\}, \ldots, \{\pi(i_n), \pi(j_n)\}\}$$

$\mathcal{B}_{2n}$ = **algebra** of all complex $\mathcal{M}_{2n} \times \mathcal{M}_{2n}$ matrices commuting with the S_{2n} action on $\mathcal{M}_{2n}$, i.e., for a $\mathcal{M}_{2n} \times \mathcal{M}_{2n}$ matrix N

$N \in \mathcal{B}_{2n}$ iff N is constant on the **S_{2n}-orbits of $\mathcal{M}_{2n} \times \mathcal{M}_{2n}$**

S_{2n}-orbits of $\mathcal{M}_{2n} \times \mathcal{M}_{2n}$

Even partitions of $2n$ are partitions of $2n$ with all parts even. They look like $2\lambda = (2\lambda_1, \ldots, 2\lambda_k)$ where $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

Let $A, B \in \mathcal{M}_{2n}$. Think of A as the **red** edges and B as the **blue** edges. Consider $A \cup B$. Each component is an even cycle and the number of vertices of the components gives an **even** partition $d(A, B) \vdash 2n$.

Lemma (A, B), $(C, D) \in \mathcal{M}_{2n} \times \mathcal{M}_{2n}$ are in the same S_{2n}-orbit if and only if $d(A, B) = d(C, D)$. Moreover, (A, B), (B, A) are in the same S_{2n}-orbit, for all $(A, B) \in \mathcal{M}_{2n} \times \mathcal{M}_{2n}$.

$\{N(2\mu) : \mu \vdash n\}$ is the **orbital basis** of $\mathcal{B}_{2n}$, where $N(2\mu) \in \mathcal{B}_{2n}$ is the symmetric $0,1$ matrix with entry in row A, column B equal to 1 iff $d(A,B) = 2\mu$. So $\dim(\mathcal{B}_{2n}) = p(n)$, the **number of partitions of n**.

$\mathcal{B}_{2n}$ is an algebra of symmetric matrices, so is commutative and thus the S_{2n}-module $\mathbb{C}[\mathcal{M}_{2n}]$ is multiplicity free. $\mathcal{B}_{2n}$ is called the **Bose-Mesner algebra** of the **perfect matching association scheme**.

What are the $p(n)$ (common) eigenspaces of $\mathcal{B}_{2n}$? For $\lambda \vdash n$, let V^{λ} denote the S_n-irreducible parametrized by λ.

Fundamental Theorem $\mathbb{C}[\mathcal{M}_{2n}] \cong \oplus_{\lambda \vdash n} V^{2\lambda}$, as S_{2n}-modules.

Many proofs. Short proof in **James and Kerber** and **Saxl**.

So both the orbital basis and the eigenspaces of $\mathcal{B}_{2n}$ are indexed by even partitions of $2n$. For $\lambda, \mu \vdash n$, define

$\theta^{2\lambda}_{2\mu} =$ **eigenvalue** of $N(2\mu)$ on $V^{2\lambda}$ (*easily shown to be an integer*).

We shall now describe our main results: one on eigenvalues and one on eigenvectors.

Notational convention Let $\mu \vdash m$ with all parts ≥ 2. For $n \geq m$ we can consider μ as a partition of n by adding 1 (the trivial part) $n - m$ times. We shall write $(\mu, 1^{n-m})$ for this partition but **pronounce** it as μ'.

Eigenvalues of $\mathcal{B}_{2n}$

Motivation In studying a natural Markov chain on perfect matchings,

Diaconis and Holmes considered the matrix

$$N((4, 2^{n-2})) = N(2(2, 1^{n-2}))$$

and gave a **universal formula** for $\theta^{2\lambda}_{2(2,1^{n-2})}, \ \lambda \vdash n$.

We generalize this as follows:

Fix a partition $\mu \vdash m$ with all parts ≥ 2. We give an algorithm that

produces a **universal formula** for $\theta^{2\lambda}_{2(\mu,1^{n-m})}, \ \lambda \vdash n \geq m$.

Eigenvectors of $\mathcal{B}_{2n}$

Motivation During the course of giving an algebraic proof of the Erdős-Ko-Rado theorem on intersecting families of perfect matchings, **Godsil and Meagher** write down an eigenvector (using a quotient argument) belonging to the eigenspace $V^{(2n-2,2)}$.

We generalize this by giving an inductive procedure to write down an eigenvector in each of the eigenspaces.

Rest of the talk Universal formula for $\theta^{2\lambda}_{2(\mu,1^{n-m})}$.

For $\lambda, \mu \vdash n$,

C_μ = **conjugacy class** in S_n of cycle type μ and $c_\mu = \sum_{\pi \in C_\mu} \pi \in \mathbb{C}[S_n]$,

χ^λ = character of V^λ, $\chi^\lambda_\mu = \chi^\lambda(\pi)$, $\pi \in C_\mu$,

ϕ^λ_μ = **eigenvalue** of c_μ on $V^\lambda = \frac{|C_\mu|\chi^\lambda_\mu}{\dim(V^\lambda)}$,

ϕ^λ_μ is called a **central character** (*can easily be shown to be an integer*).

$\Lambda =$ **Symmetric functions** in $\{x_1, x_2, x_3, \ldots\}$ with coefficients in $\mathbb{Q}[t]$.

Power sum symmetric function: $p_0 = 1$ and $p_n = \sum_i x_i^n, \; n \geq 1$.

$p_\lambda \; = \; p_{\lambda_1} p_{\lambda_2} \cdots$ if $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$ $(=$ all partitions $)$.

The set $\{p_\lambda \; : \; \lambda \in \mathcal{P}\}$ is a $\mathbb{Q}[t]$-module basis of Λ.

The **content** $c(b)$ of a box b of a Young diagram is

$$y\text{-coordinate of box } b - x\text{-coordinate of box } b.$$

Contents in the first row are $0, 1, 2, \ldots$, second row are $-1, 0, 1, 2, \ldots$ and so on.

Given $f \in \Lambda$ and $\lambda \in \mathcal{Y}_n$ (= Young diagrams with n boxes) we define the **content evaluation** $f(c(\lambda))$ to be the rational number obtained from f by setting $t = n$, $x_i = 0$ for $i > n$, and $\{x_1, \ldots, x_n\}$ to the multiset of the contents of the n boxes of λ.

Examples of universal formula

Frobenius $\phi^{\lambda}_{(2,1^{n-2})} = p_1(c(\lambda)) =$ Sum of all contents , $\lambda \in \mathcal{Y}_n$.

Ingram $\phi^{\lambda}_{(3,1^{n-3})} = (p_2 - \frac{t(t-1)}{2})(c(\lambda))$

$$= \text{ Sum of squares of all contents } - \frac{n(n-1)}{2}, \ \lambda \in \mathcal{Y}_n.$$

There is a vast generalization. Let $\mathcal{P}(2)$ denote the set of partitions with all parts ≥ 2.

Thm. 1 (Corteel, Goupil, and Schaeffer (2004) and Garsia (2003))

For each $\mu \in \mathcal{P}(2)$ there is an algorithm to compute a symmetric

function $W_\mu \in \Lambda$ such that

(i) $\{W_\mu \ : \ \mu \in \mathcal{P}(2)\}$ is a $\mathbb{Q}[t]$-module basis of Λ.

(ii) Let $\mu \in \mathcal{P}(2)$ with $|\mu| = m$. Let $\lambda \in \mathcal{P}$ with $|\lambda| = n \geq m$. We have

$$W_\mu(c(\lambda)) \quad = \quad \phi^\lambda_{(\mu, 1^{n-m})}.$$

Example $(|\mu| \leq 4)$. This gives the character tables of $S_1, \ldots, S_4$ and the first five characters of S_n, $n \geq 5$.

$W_\emptyset = 1$ $\quad W_2 = p_1$ $\quad W_3 = p_2 - \frac{t(t-1)}{2}$

$W_{2,2} = \frac{p_1^2}{2} - \frac{3p_2}{2} + \frac{t(t-1)}{2}$ $\quad W_4 = p_3 - (2t-3)p_1$

Thm. 2 For each $\mu \in \mathcal{P}(2)$ there is an algorithm to compute a symmetric function $E_\mu \in \Lambda$ such that

(i) $\{E_\mu \;:\; \mu \in \mathcal{P}(2)\}$ is a $\mathbb{Q}[t]$-module basis of Λ.

(ii) Let $\mu \in \mathcal{P}(2)$ with $|\mu| = m$. Let $\lambda \in \mathcal{P}$ with $|\lambda| = n \geq m$. We have

$$E_\mu(c(2\lambda)) \quad = \quad \theta^{2\lambda}_{2(\mu,1^{n-m})}.$$

Example $(|\mu| \leq 4)$ This gives the eigenvalues of $\mathcal{B}_2, \ldots, \mathcal{B}_8$ and the first five eigenvalues of $\mathcal{B}_{2n},\ n \geq 5$. $E_\emptyset = 1$ and

$$E_2 = \frac{p_1}{2} - \frac{t}{4} \qquad E_3 = \frac{p_2}{2} - p_1 + \frac{3t - t^2}{4}$$

$$E_{2,2} = \frac{p_1^2}{8} - \frac{3p_2}{4} + \frac{(10-t)p_1}{8} + \frac{9t^2 - 24t}{32} \qquad E_4 = \frac{p_3}{2} - \frac{9p_2}{4} + \frac{(11-2t)p_1}{2} + \frac{8t^2 - 23t}{8}$$

Two ingredients in the **Proof of Theorem 2**: Theorem 1 **plus** a **combinatorial algorithm** that, starting with the central characters of $S_2, S_4, \ldots, S_{2n}$ produces the eigenvalues of $\mathcal{B}_2, \mathcal{B}_4, \ldots, \mathcal{B}_{2n}$ by solving linear equations of size at most $p(n-1) \times p(n-1)$.

Moreover, given $\{W_\mu \mid |\mu| \leq m\}$ this algorithm produces $\{E_\mu \mid |\mu| \leq m\}$ by solving linear equations of size $p(m-1) \times p(m-1)$.

Ongoing project **Garsia's** paper contains a list of the symmetric functions W_μ for $|\mu| \leq 8$. Since $p(7) = 15$ the algorithm above needs to solve equations of size at most 15×15 to produce E_μ for $|\mu| \leq 8$. This seems **feasible**, yielding eigenvalues of $\mathcal{B}_2, \ldots, \mathcal{B}_{16}$.

THANK YOU