Non-commutative association schemes of rank 6

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Known results

1. Y. Asaba and A. Hanaki, A construction of integral standard generalized table algebras from parameters of projective geometries, *Israel J. Math.*, **194**, (2013), 395-408.
2. A. Hanaki and P.-H. Zieschang, on imprimitive noncommutative association schemes of order 6, *Comm. Algebra*, **42** (3), (2014), 1151-1199.
3. M. Yoshikawa, On noncommutative integral standard table algebras in dimension 6, *Comm. Algebra*, **42** (2014), 2046-2060.
4. B. Drabkin and C. French, On a class of noncommutative imprimitive association schemes of rank 6, *Comm. Algebra*, **43** (9), (2015), 4008-4041.
5. C. French and P.-H. Zieschang, On the normal structure of noncommutative association schemes of rank 6, *Comm. Algebra*, **44** (3), 2016, 1143-1170.

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In all those papers it was assumed that the scheme is imprimitive.

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If $R, S \subseteq X^2$ are binary relations on a finite set X, then

1. $R(x) := \{y \in X \mid (x, y) \in R\}$;
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If $\mathbb{F}$ is a field, then

1. $M_X(\mathbb{F})$ is the matrix algebra;
2. I_X is the identity matrix;
3. J_X is all one matrix;
4. $^\top$ is is matrix transposition;

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3. for any triple $i, j, k \in \{0, \dots, d\}$ and any pair $(x, y) \in R_k$ the intersection number $p_{ij}^k := |R_i(x) \cap R_{j'}(y)|$ depends only on i, j, k.

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3. $|\mathcal{R}|$ - the rank of $\mathfrak{X}$.

Adjacency (BM-) algebra of a scheme

Theorem

Let A_i be the adjacency matrix of the basic graph (X, R_i). Then the linear span $\mathcal{A}_{\mathbb{F}} := \langle A_0, ..., A_d \rangle$ is a subalgebra of the matrix algebra $M_X(\mathbb{F})$. Moreover $I_X, J_X \in \mathcal{A}_{\mathbb{F}}, \mathcal{A}_{\mathbb{F}}^{\top} = \mathcal{A}_{\mathbb{F}}$ and

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^{k} A_k.$$

$\mathcal{A}_{\mathbb{F}}$ is called the adjacency / Bose-Mesner algebra of $\mathfrak{X}$. The basis $A_0, ..., A_d$ is called the standard basis of $\mathcal{A}_{\mathbb{F}}$.

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Proposition

A symmetric scheme is commutative $\Rightarrow$
A non-commutative scheme contains at least one pair of anti-symmetric relations.

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Let $\mathfrak{X} = (X, \mathcal{R} = \{R_i\}_{i=0}^d)$ be an association scheme and $\mathcal{A}_{\mathbb{F}} = \langle A_0, \ldots, A_d \rangle$ its BM-algebra, $\mathrm{char}(\mathbb{F}) = 0$. The following conditions are equivalent

(a) $\mathfrak{X}$ is imprimitive;

(b) $\exists\ I \subset \{0, \ldots, d\}$ s.t. $1 < |I| \leq d$ and $R_I := \bigcup_{i \in I} R_i$ is an equivalence relation on X;

(c) $\exists\ I \subset \{0, \ldots, d\}$ s.t. $I' = I$ and $\langle A_i \rangle_{i \in I}$ is a subalgebra of $\mathcal{A}_{\mathbb{F}}, \mathrm{char}(\mathbb{F}) = 0$.

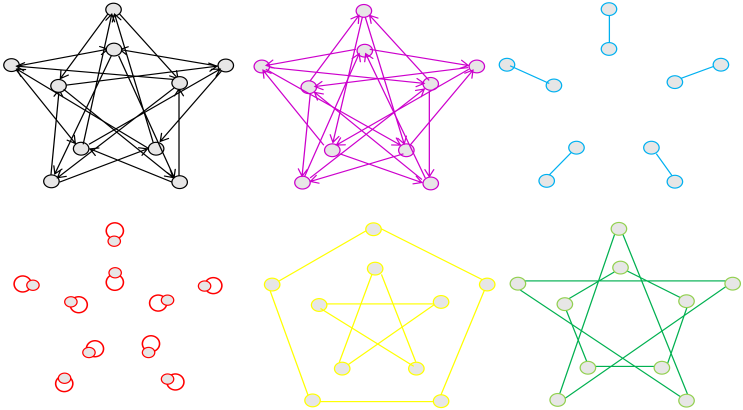
The subset $\{R_i\}_{i \in I}$ is called a closed subset of $\mathcal{R}$.

A concrete example (M. Klin and A.Woldar)

$$A(\mathfrak{X}) = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 3 & 4 & 5 & 5 & 4 \\ 1 & 0 & 1 & 2 & 2 & 4 & 3 & 4 & 5 & 5 \\ 2 & 1 & 0 & 1 & 2 & 5 & 4 & 3 & 4 & 5 \\ 2 & 2 & 1 & 0 & 1 & 5 & 5 & 4 & 3 & 4 \\ 1 & 2 & 2 & 1 & 0 & 4 & 5 & 5 & 4 & 3 \\ 3 & 5 & 4 & 4 & 5 & 0 & 2 & 1 & 1 & 2 \\ 5 & 3 & 5 & 4 & 4 & 2 & 0 & 2 & 1 & 1 \\ 4 & 5 & 3 & 5 & 4 & 1 & 2 & 0 & 2 & 1 \\ 4 & 4 & 5 & 3 & 5 & 1 & 1 & 2 & 0 & 2 \\ 5 & 4 & 4 & 5 & 3 & 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

Example (the basic graphs)

back



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4. the others.

Schemes coming from groups

Let $G \leq \mathrm{Sym}(X)$ be a transitive permutation group,
$\Pi : \mathbb{F}[G] \to M_X(\mathbb{F})$ corresponding representation of G,
$R_0 = I_X, R_1, ..., R_d$ be the complete set of 2-orbits (orbitals) of G.

Proposition

The set of relations $R_i, i = 0, ..., d$ form an association scheme on X. Its BM-algebra coincides with $C_{M_X(\mathbb{F})}(\Pi(\mathbb{F}[G]))$. Association schemes of this type are called Schurian.

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Example

If G acts regularly on X, then the relations R_i are permutations of X which form a regular permutation subgroup of $\mathrm{Sym}(X)$ isomorphic to G. All basic relations of this scheme are thin (have valency one).

The BM-algebra of this scheme is isomorphic to $\mathbb{F}[G]$.

Class merging (fusion and fission schemes)

Definition

Let $\mathfrak{X} = (X, \mathcal{R} = \{R_i\}_{i=0}^{d})$ and $\mathfrak{X}' = (X, \mathcal{R}' = \{R_i'\}_{i=0}^{d'})$ be two association schemes with the same point set X. We say that $\mathfrak{X}'$ is a fusion of $\mathfrak{X}$ (or $\mathfrak{X}$ is a fission of $\mathfrak{X}'$) iff each R_i' is a union of some R_j.

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Proposition

$\mathfrak{X}' = (X, \mathcal{R}' = \{R_i'\}_{i=0}^{d'})$ is a fusion of $\mathfrak{X} = (X, \mathcal{R} = \{R_i\}_{i=0}^{d})$ iff there exists a partition $T_0, \ldots, T_{d'}$ of $\{0, 1, \ldots, d\}$ such that

1. $T_0 = \{0\}$;
2. $\forall_i \; \exists \, j \;\; T_i' = T_j$;
3. $\forall_i \;\; R_i' = \bigcup_{j \in T_i} R_i$.

Example

Flag scheme of a projective plane

Let $\Pi = (P, L)$ be a projective plane of order n. Denote by $\mathcal{F}$ the set of flags (p, ℓ) of the plane Π. Define two relations on $\mathcal{F}$ as following

$$\begin{array}{rcl} S & := & \{((p_1, \ell_1), (p_2, \ell_2)) \,|\, p_1 = p_2, \ell_1 \neq \ell_2\}, \\ T & := & \{((p_1, \ell_1), (p_2, \ell_2)) \,|\, P_1 \neq p_2, \ell_1 = \ell_2\}. \end{array}$$

Then the relations $1_{\mathcal{F}}, S, T, ST, TS, TST$ form an association scheme of rank 6 on $\mathcal{F}$ called the flag scheme of a projective plane.

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The flag scheme is non-commutative and imprimitive.

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In the second case the basic graphs form a complementary pair of strongly regular graphs. The parameters are completely determined by $p^0_{11}, p^1_{11}, p^2_{11}$.

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In both cases the schemes are commutative.

BM-algebra of an association scheme.

Theorem (B. Weisfeiler & A. Leman, D. Higman)

Let $\mathfrak{X} = (X, \mathcal{R})$ be a scheme. It's BM-algebra $\mathcal{A}_{\mathbb{F}}$ is semisimple if $\mathrm{char}(\mathbb{F}) = 0$. If, in addition, $\mathbb{F}$ is algebraically closed, then

$$\mathcal{A}_{\mathbb{F}} \cong \oplus_{i=0}^{k} M_{m_i}(\mathbb{F}), \text{ with } m_0 = 1.$$

In particular, $|\mathcal{R}| = \sum_{i=0}^{k} m_i^2$.

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Corollary

A BM-algebra of a non-commutative rank six scheme over algebraically closed field $\mathbb{F}$ of characteristic zero is isomorphic to $\mathbb{F} \oplus \mathbb{F} \oplus M_2(\mathbb{F})$.

Non-commutative association schemes of rank 6

Theorem

Let $\mathfrak{X} = (X, \{R_0, ..., R_5\})$ be a non-commutative rank six association scheme of order n. Let $\mathcal{A} := \langle A_0, ..., A_5 \rangle_{\mathbb{R}}$ be BM-algebra of $\mathfrak{X}$ defined over the reals. Then

1. $\exists$ an algebra isomorphism $\Theta : \mathcal{A} \to \mathbb{R} \oplus \mathbb{R} \oplus M_2(\mathbb{R})$;
2. $\Theta(A^\top) = \Theta(A)^\top$;
3. $A_i^\top = A_i$ if $0 \leq i \leq 3$ and $A_4^\top = A_5$.

Thus $\Theta(A) = (\delta(A), \phi(A), B(A))$ where δ, ϕ and B are three absolutely irreducible real representations of $\mathcal{A}$.
In what follows δ is a degree map ($\delta(A_i)$ equals the valency of R_i).

The image of the standard basis

The elements $b_i := \Theta(A_i) = (\delta_i, \phi_i, B_i), i = 0, ..., 5$ form a basis of $M_{1,1,2}(\mathbf{R}) := \mathbf{R} \oplus \mathbf{R} \oplus M_2(\mathbf{R})$ s.t.

1. $b_0 = (1, 1, I_2)$ is the identity of $M_{1,1,2}(\mathbf{R})$;
2. $b_1^\top = b_1, b_2^\top = b_2, b_3^\top = b_3, b_4^\top = b_5, b_5^\top = b_4$;
3. the structure constants p_{ij}^k of the basis **B** are non-negative integers;
4. $p_{ij}^0 = 0$ if $b_i^\top \neq b_j$ and δ_i otherwise.

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A basis satisfying 1,2,4 is called a reality basis (H.Blau) of $\mathcal{A}$. The number $\delta_0 + ... + \delta_5$ is called degree/order of **B**.

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A reality basis is called a table basis (Z. Arad and H. Blau) if p_{ij}^k are non-negative reals.

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4. $p_{ij}^0 = 0$ if $b_i^\top \neq b_j$ and δ_i otherwise.

A basis satisfying 1,2,4 is called a reality basis (H.Blau) of $\mathcal{A}$. The number $\delta_0 + ... + \delta_5$ is called degree/order of **B**.

A reality basis is called a table basis (Z. Arad and H. Blau) if p_{ij}^k are non-negative reals.

A reality basis is called integral iff all p_{ij}^k are integers.

Enumeration of integral table bases

Definition

Two table bases $\mathbf{B}$ and $\tilde{\mathbf{B}}$ of $M_{1,1,2}(\mathbb{R})$ are equivalent if there exists a $^{\top}$-permutable automorphism φ of $M_{1,1,2}(\mathbb{R})$ such that $\mathbf{B}^{\varphi} = \tilde{\mathbf{B}}$.

Two tables bases $\mathbf{B}, \tilde{\mathbf{B}}$ of $\mathcal{A}$ are equivalent iff there exists a permutation φ of $\{0, 1, ..., d\}$ which commutes with $^{\top}$ and satisfies s.t. $\tilde{p}_{ij}^{k} = p_{\varphi(i),\varphi(j)}^{\varphi(k)}$ for all i, j, k.

Problem

Given a number n, find all integral table bases of order n (up to equivalency) of the algebra $M_{1,1,2}(\mathbb{R})$.

Character Table

The algebra $\mathcal{A} \cong M_{1,1,2}(\mathbb{R})$ has three irreducible characters δ, ϕ and $\chi(A) := \mathrm{tr}(B(A))$.
The standard character of $\mathcal{A}$: $\tau(A) := \mathrm{tr}(A)$.
$\tau = \delta + m_\phi \phi + m_\chi \chi$, where m_ϕ and m_χ are the multiplicities.

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The character table

	b_0	b_1	b_2	b_3	b_4	b_5
δ	1	δ_1	δ_2	δ_3	δ_4	$\delta_5 = \delta_4$
ϕ	1	ϕ_1	ϕ_2	ϕ_3	ϕ_4	$\phi_5 = \phi_4$
χ	2	χ_1	χ_2	χ_3	χ_4	$\chi_5 = \chi_4$

Orthogonality relations

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Row orthogonality

$$
\begin{aligned}
1+\delta_1+\delta_2+\delta_3+2\delta_4 &= n \\
1+\frac{\phi_1^2}{\delta_1}+\frac{\phi_2^2}{\delta_2}+\frac{\phi_3^2}{\delta_3}+2\frac{\phi_4^2}{\delta_4} &= \frac{n}{m_\phi} \\
4+\frac{\chi_1^2}{\delta_1}+\frac{\chi_2^2}{\delta_2}+\frac{\chi_3^2}{\delta_3}+2\frac{\chi_4^2}{\delta_4} &= 2\frac{n}{m_\chi} \\
1+\phi_1+\phi_2+\phi_3+2\phi_4 &= 0 \\
2+\chi_1+\chi_2+\chi_3+2\chi_4 &= 0 \\
2+\frac{\phi_1\chi_1}{\delta_1}+\frac{\phi_2\chi_2}{\delta_2}+\frac{\phi_3\chi_3}{\delta_3}+2\frac{\phi_4\chi_4}{\delta_4} &= 0
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1+\phi_1+\phi_2+\phi_3+2\phi_4 &= 0\\
2+\chi_1+\chi_2+\chi_3+2\chi_4 &= 0\\
2+\frac{\phi_1\chi_1}{\delta_1}+\frac{\phi_2\chi_2}{\delta_2}+\frac{\phi_3\chi_3}{\delta_3}+2\frac{\phi_4\chi_4}{\delta_4} &= 0
\end{aligned}$$

Column orthogonality

$$\begin{aligned}
1+m_\phi+2m_\chi &= n;\\
\forall_{i=1,\dots,5}\quad \delta_i+m_\phi\phi_i+m_\chi\chi_i &= 0
\end{aligned}$$

Necessary conditions

Proposition

If $\mathbf{B}$ is an integral table basis, then $\delta_i, \phi_i, \chi_i, i = 1, \dots 4$ are integers and

1. $\forall_i \; \delta_i > 0$;
2. $|\phi_i| \leq \delta_i, |\chi_i| \leq 2\delta_i$.

Proposition

If the table basis comes from a scheme, then the multiplicities m_ϕ, m_χ are positive integers.

Proposition

The set $\{\phi_i/\delta_i\}_{i=1}^4$ contains at least two numbers. If it contains exactly two numbers, then the center of $\mathcal{A}$ is a BM-algebra of a rank three fusion scheme $\mathfrak{X}$.

Main Results

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The numbers $\delta_1, \delta_2, \delta_3, \delta_4, \phi_1, \phi_2, \phi_3$ determine the character table T of $(\mathcal{A}, \mathbf{B})$ uniquely.

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Theorem

Given the character table T of $(\mathcal{A}, \mathbf{B})$, there exists a unique (up to an equivalency) reality basis $\mathbf{B}$ with that character table. In particular, the numbers $\delta_1, \delta_2, \delta_3, \delta_4, \phi_1, \phi_2, \phi_3$ determine the structure constants of the BM-algebra $\mathcal{A}$.

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We have enumerated all tuples $\delta_1, \delta_2, \delta_3, \delta_4, \phi_1, \phi_2, \phi_3$ which provide a character table T with integral entries.
Then for each character table T we construct a table basis $\mathbf{B}$ and check whether the structure constants w.r.t. $\mathbf{B}$ are non-negative integers.

Enumeration results

We obtained all feasible parameters of non-commutative schemes of rank six up to order 150. Among them four parameter sets for primitive schemes were found.

N	n	δ, ϕ	(m_ϕ, m_χ)
1	81	$[10, 10, 20, 20], [1, 1, -7, 2]$	$(20, 30)$
2	96	$[19, 19, 19, 19], [-5, -5, 3, 3]$	$(19, 38)$
3	96	$[19, 19, 19, 19], [3, 3, 3, -5]$	$(19, 38)$
4	120	$[17, 17, 51, 17], [-3, -3, 3, 1]$	$(51, 34)$

Primitive rank six schemes

Theorem

There is only one feasible parameter set corresponding to primitive non-commutative rank six scheme of order ≤ 150. It has order 81 and the valencies $1, 10, 10, 20, 20, 20$.

Proof. The second and third algebras have rank three fusion scheme with degrees $1, 38, 57$. According to Brouwer's table an SRG with such parameters doesn't exists.
The last algebra violates the condition $p_{ij}^{i}\delta_i \equiv 0 (\bmod 2)$ whenever $i' = i, j' = j$. □

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Find an example of a primitive non-commutative association scheme of rank 6 (if it exists).

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Lemma (Munemasa)

There is no schurian primitive association schemes of rank 6 with less than 1600 points.

Schurian case

Let $G \leq \mathrm{Sym}(X)$ be a transitive permutation group, $\mathfrak{X} = (X, \mathcal{R})$ its 2-orbit scheme.

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Classify all transitive (primitive) permutation groups $G \leq \mathrm{Sym}(X)$ satisfying $1_{G_x}^G = 1_G + \alpha + 2\beta$.

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Theorem (M. Conder and V. Jones)

If $\mathfrak{X}$ has rank 6 and admits two closed subsets $I, J \subseteq \{0, 1, ..., 5\}$ such that $R_I R_J \neq R_J R_I$, then either $\mathfrak{X}$ is a thin scheme isomorphic to S_3 or $\mathfrak{X}$ is a flag scheme of a Desarguesian plane.

Central rank of a permutation group

Let $G \leq \mathrm{Sym}(X)$ be a permutation group,
$\Pi : \mathbb{C}[G] \to M_X(\mathbb{C})$ is a corresponding representation of G,
$\mathfrak{X} = (X, \{R_0, R_1, ..., R_d\})$ is the 2-orbit scheme of G,
$\mathcal{A}$ is the BM-algebra of $\mathfrak{X}$.

Theorem (H. Wielandt)

$\mathcal{A} \cap \Pi(\mathbb{C}[G]) = Z(\mathcal{A}) = \Pi(Z(\mathbb{C}[G]))$.

In what follows we call $\dim(Z(\mathcal{A}))$ the central rank of G and denote as $c\!-\!rank(G)$.

Proposition

1. The central rank of a permutation group is equal to the number of irreducible constituents in the decomposition of Π;
2. $2 \leq c\!-\!rank(G) \leq rank(G)$ where the equality holds iff Π is multiplicity free.

Central rank of a permutation group

It is well-known that $H \leq G \implies rank(H) \geq rank(G)$.
But $H \leq G \not\Longleftarrow c\text{–}rank(H) \geq c\text{–}rank(G)$. For example, take H to be a regular subgroup of S_6 isomorphic to S_3 and G to be $\mathbb{Z}_3 \wr \mathbb{Z}_2$ in imprimitive action. Then $c\text{–}rank(H) = 3$ while $c\text{–}rank(G) = 4$.

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Proposition

Let $\mathcal{P}$ be G-invariant partition of X. Then $c\text{–}rank(G^{\mathcal{P}}) \leq c\text{–}rank(G)$.

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Theorem

If G is transitive and $c\text{–}rank(G) = 2$ then $rank(G) = 2$, that is G acts 2-transitively on X.

The result is not true if G is intransitive (for example, $c\text{–}rank(S_2 \boxplus S_2) = 2$, but $rank(S_2 \boxplus S_2) = 8$).

Primitive permutation group of central rank three

Each rank three group is a *c–rank* three group, but not versa.

Example

The group $PGL_3(q)$ acting on the flags of the projective plane has rank six and *c–rank* three .

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Theorem

If one of the groups A_n, S_n has a primitive action with central rank three, then the rank of this action is also three.

Thank you!