Quantum walks on graphs: state transfer

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Continuous-time quantum walk

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Transition matrix

$$U(t) = \exp(itA)$$

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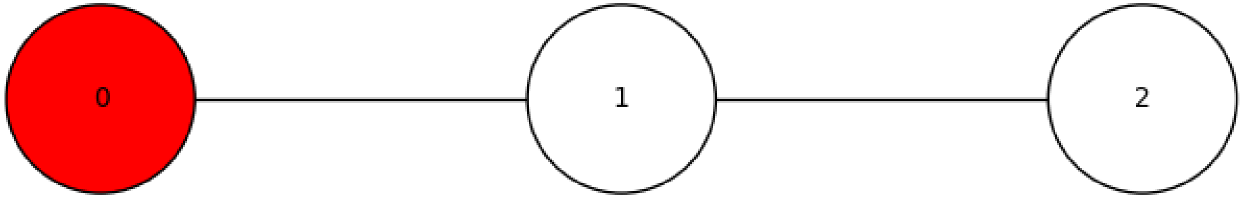
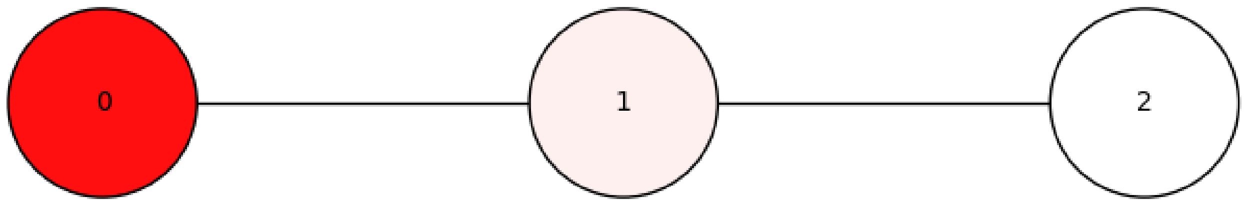
Transition matrix

$$\begin{aligned} U(t) &= \exp(itA) \\ &= I + itA - \frac{1}{2!}t^2A^2 - \frac{i}{3!}t^3A^3 + \cdots \end{aligned}$$

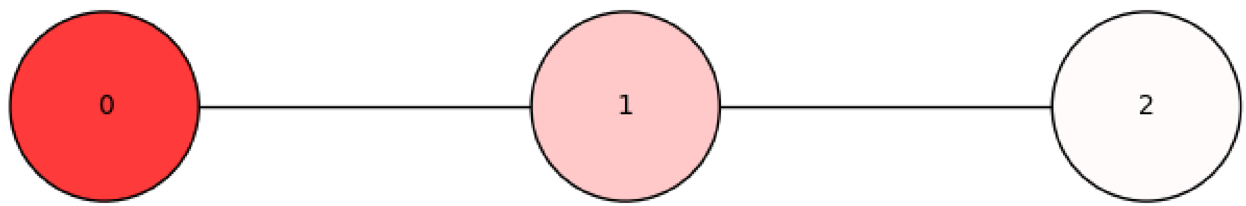


$$\begin{pmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}$$



$$\begin{pmatrix} 0.939105 & 0.059939 & 0.000956 \\ 0.059939 & 0.880122 & 0.059939 \\ 0.000956 & 0.059939 & 0.939105 \end{pmatrix}$$

Time incrementing by 0.25.



$$\begin{pmatrix} 0.774615 & 0.211014 & 0.014371 \\ 0.211014 & 0.577972 & 0.211014 \\ 0.014371 & 0.211014 & 0.774615 \end{pmatrix}$$

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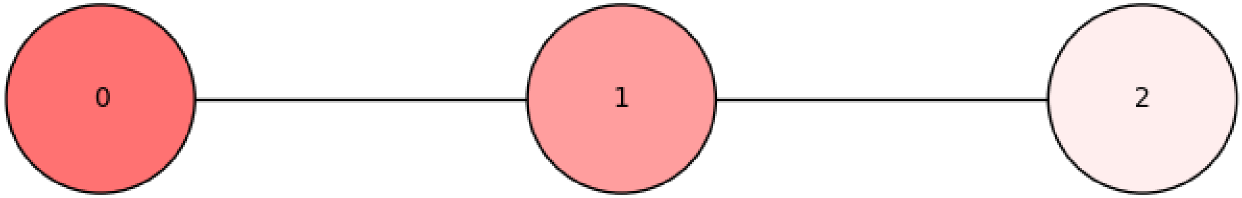

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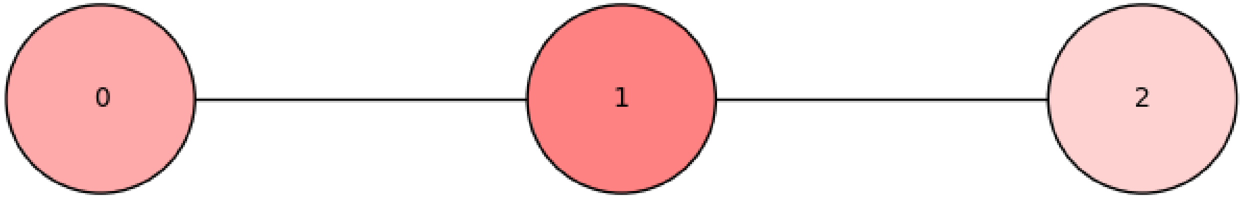

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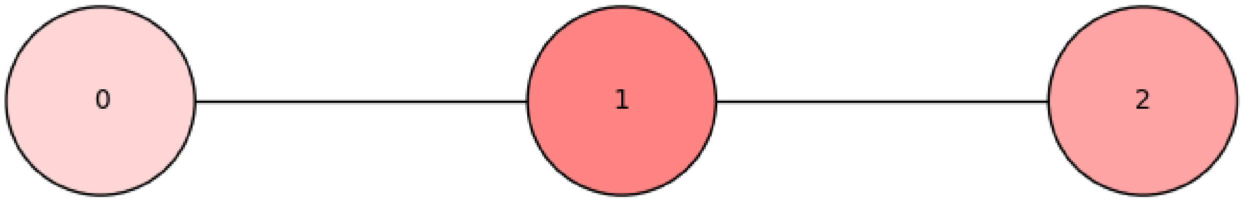

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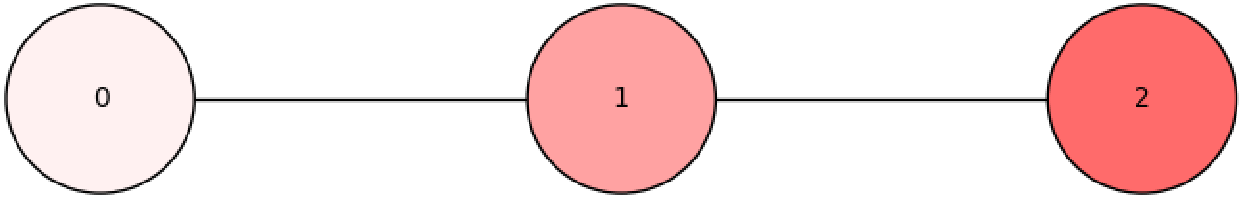

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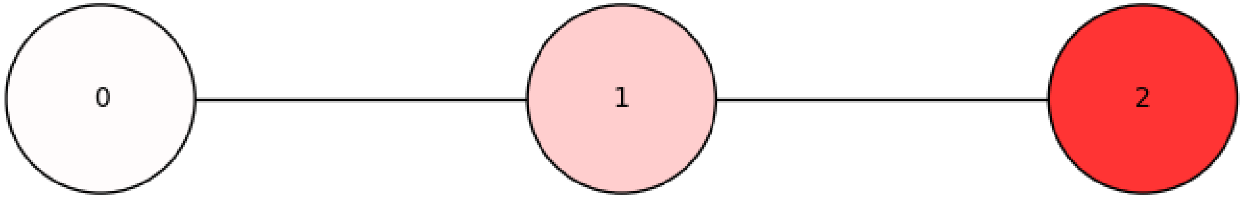

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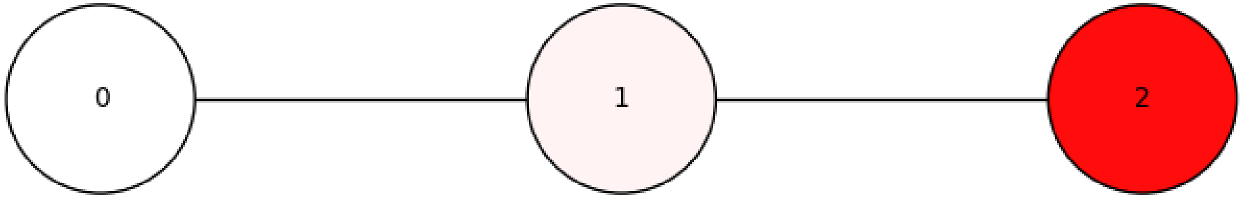

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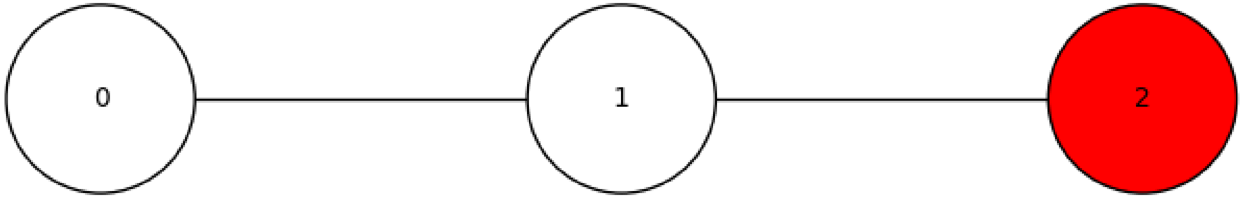

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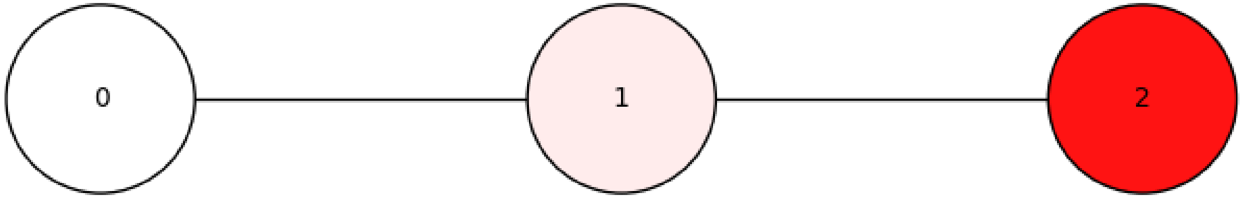

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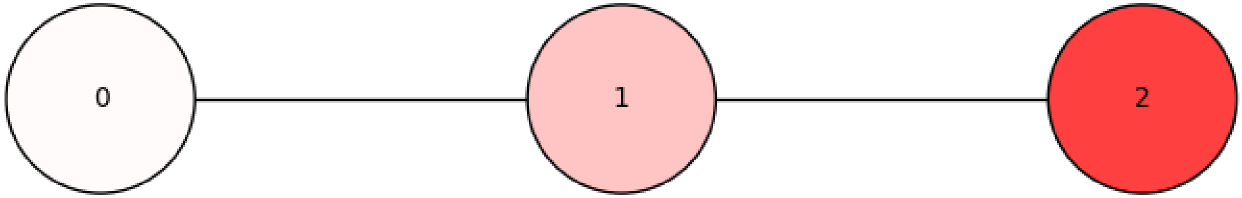

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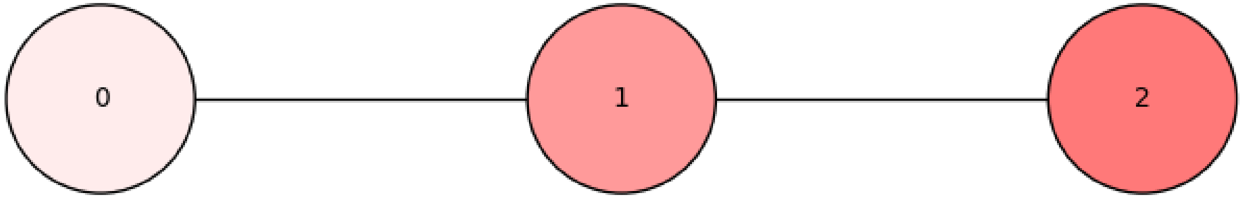

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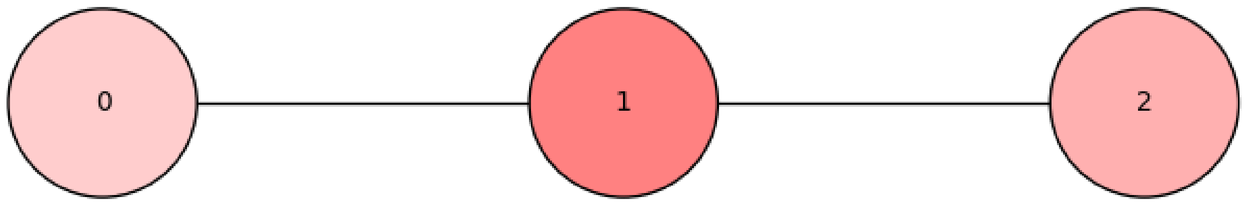

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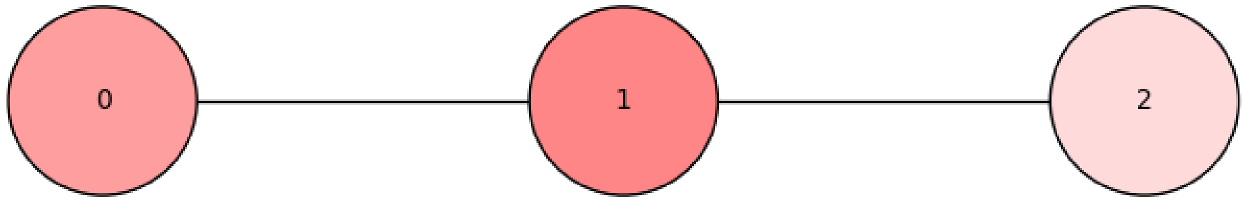

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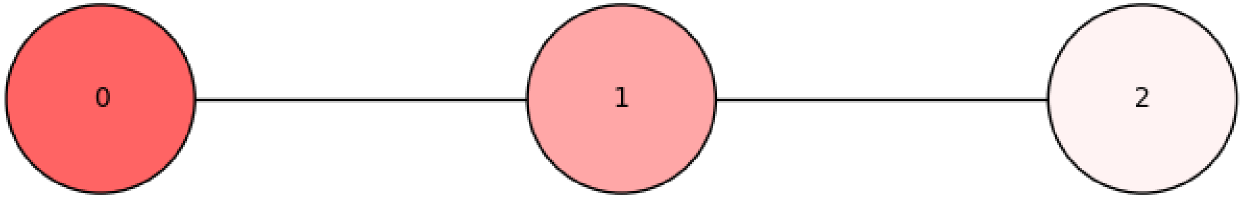

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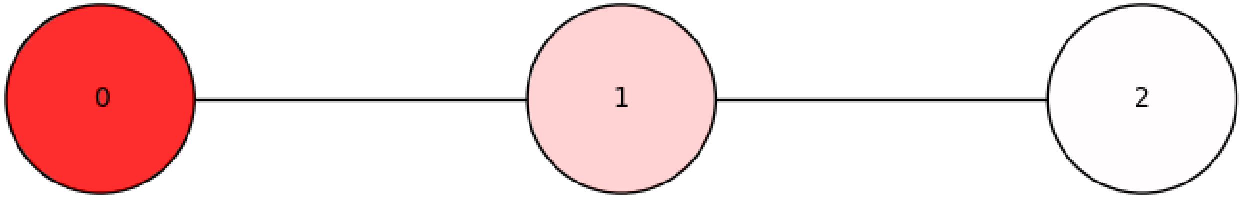

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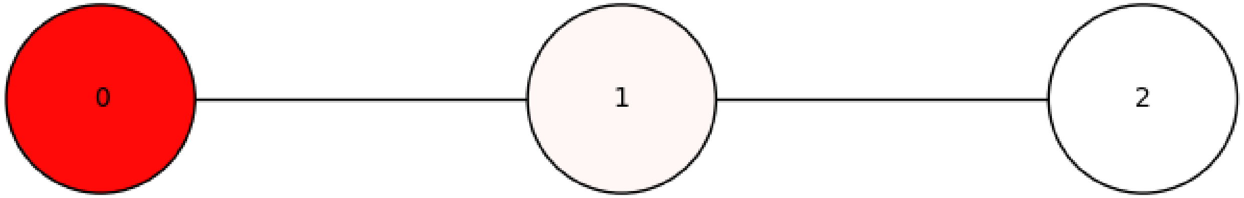

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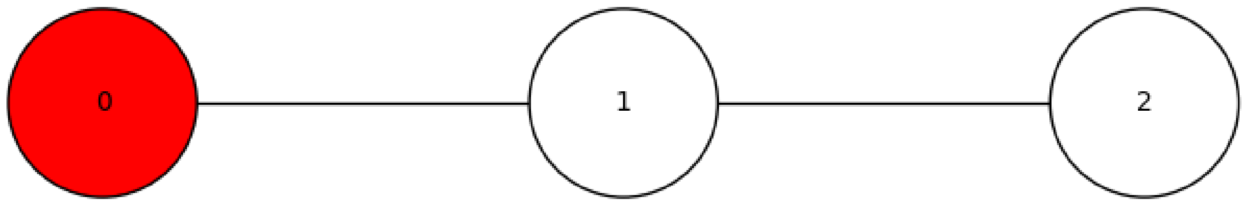

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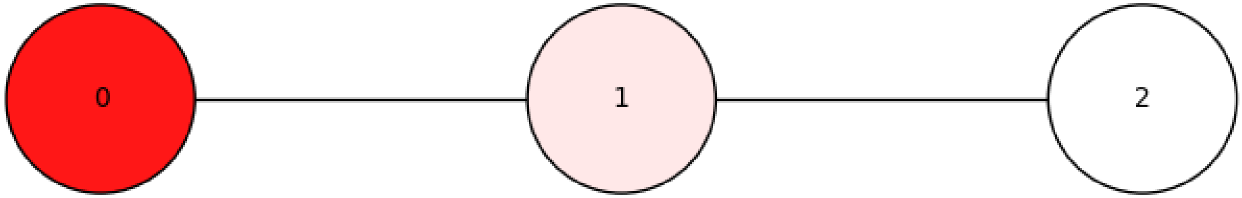

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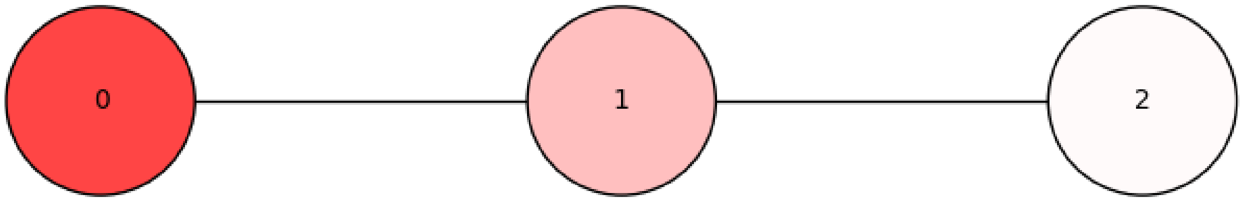

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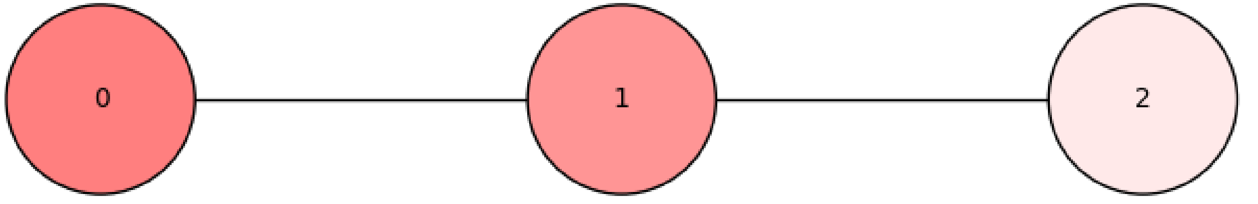

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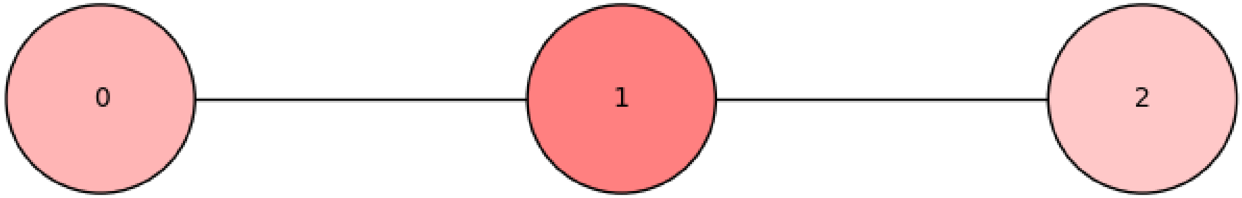

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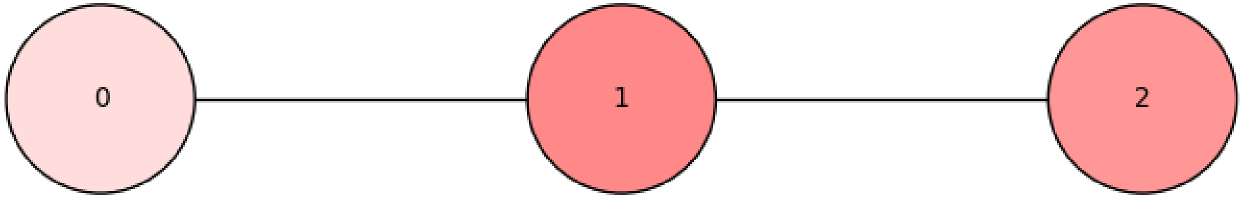

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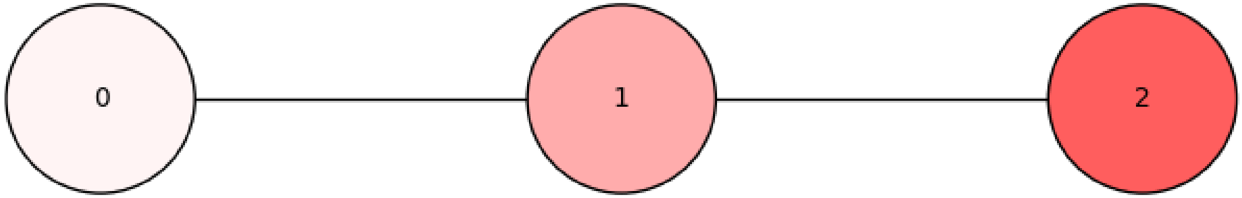

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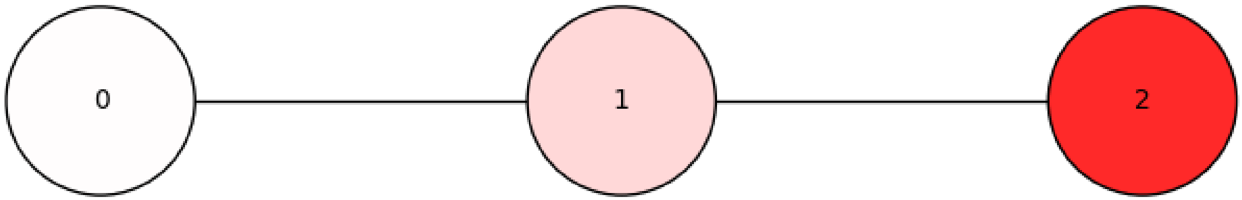

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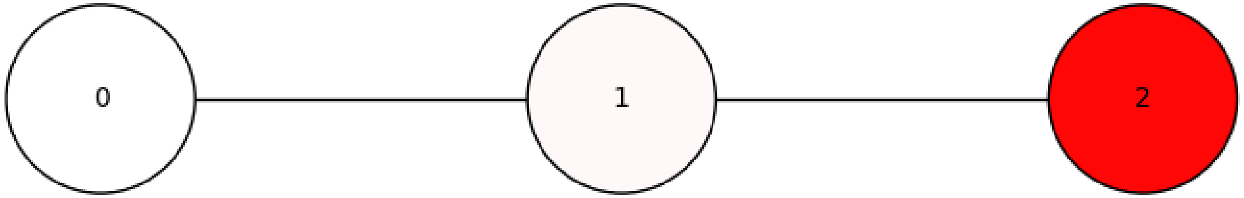

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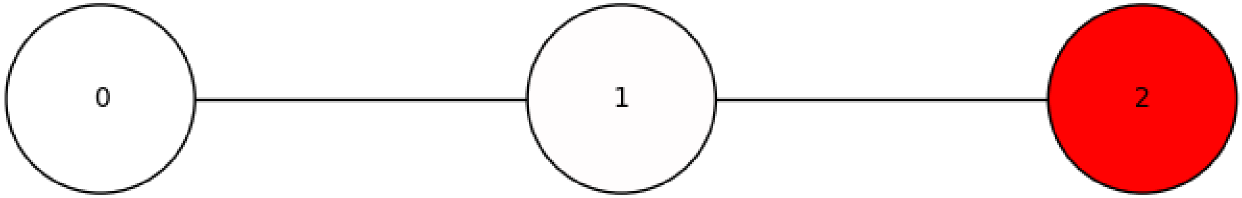

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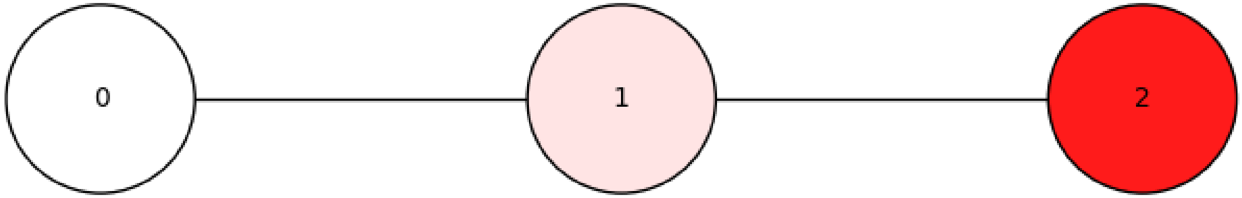

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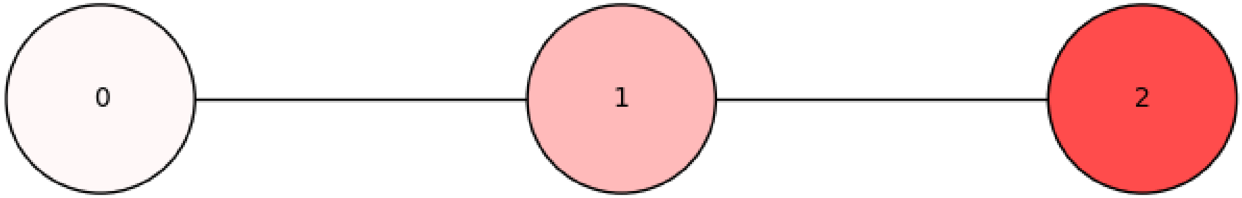

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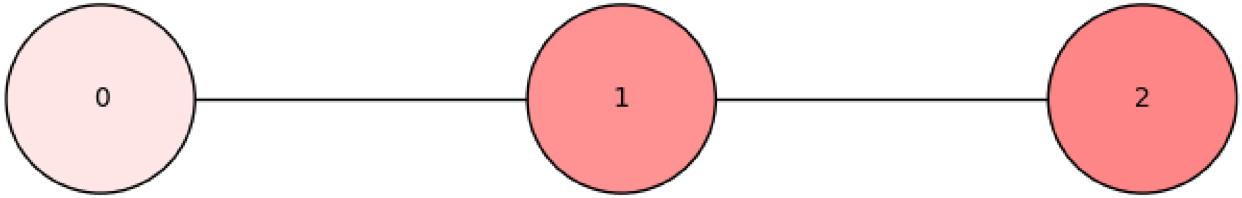

Time incrementing by 0.25.

Perfect state transfer

For P_3:

Perfect state transfer

For P_3:

$$U(t) = e^{it0} \begin{pmatrix} 0.5 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{pmatrix} + e^{it\sqrt{2}} \begin{pmatrix} 0.25 & \frac{\sqrt{2}}{4} & 0.25 \\ \frac{\sqrt{2}}{4} & 0.5 & \frac{\sqrt{2}}{4} \\ 0.25 & \frac{\sqrt{2}}{4} & 0.25 \end{pmatrix} + e^{-it\sqrt{2}} \begin{pmatrix} 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \\ -\frac{\sqrt{2}}{4} & 0.5 & -\frac{\sqrt{2}}{4} \\ 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \end{pmatrix}$$

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$$+ e^{-it\sqrt{2}}\begin{pmatrix} 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \\ -\frac{\sqrt{2}}{4} & 0.5 & -\frac{\sqrt{2}}{4} \\ 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \end{pmatrix}$$

$$U(\tfrac{\pi}{\sqrt{2}}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

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$$+ e^{-it\sqrt{2}}\begin{pmatrix} 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \\ -\frac{\sqrt{2}}{4} & 0.5 & -\frac{\sqrt{2}}{4} \\ 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \end{pmatrix}$$

$$U(\frac{\pi}{\sqrt{2}})e_0 = (-1)e_2$$

Perfect state transfer

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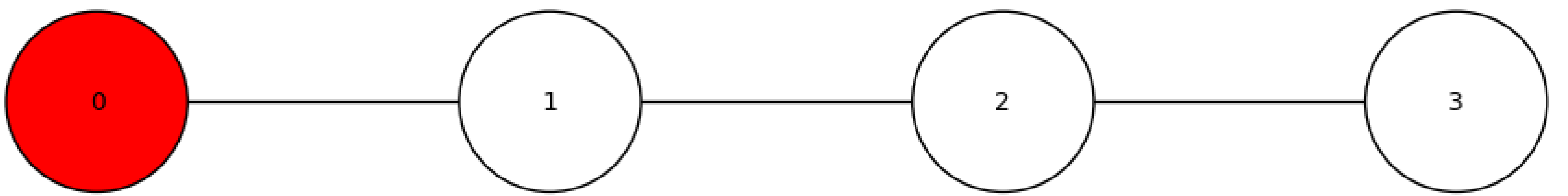
$$U(t) = e^{it0} \begin{pmatrix} 0.5 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 \\ -0.5 & 0.0 & 0.5 \end{pmatrix} + e^{it\sqrt{2}} \begin{pmatrix} 0.25 & \frac{\sqrt{2}}{4} & 0.25 \\ \frac{\sqrt{2}}{4} & 0.5 & \frac{\sqrt{2}}{4} \\ 0.25 & \frac{\sqrt{2}}{4} & 0.25 \end{pmatrix} + e^{-it\sqrt{2}} \begin{pmatrix} 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \\ -\frac{\sqrt{2}}{4} & 0.5 & -\frac{\sqrt{2}}{4} \\ 0.25 & -\frac{\sqrt{2}}{4} & 0.25 \end{pmatrix}$$

perfect state transfer from a to b: there exists τ such that

$$U(\tau)e_a = \gamma e_b$$

where $|\gamma| = 1$.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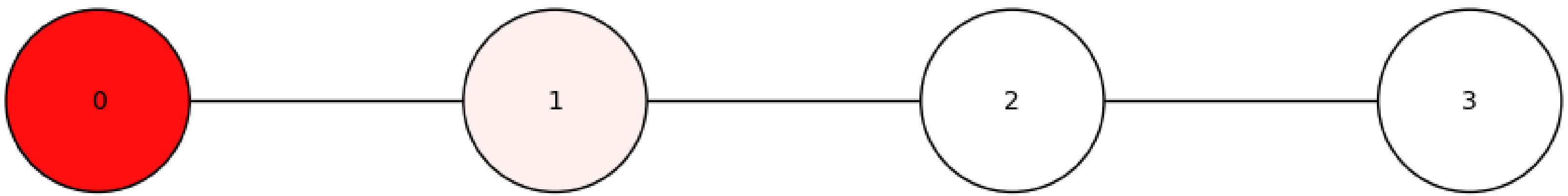

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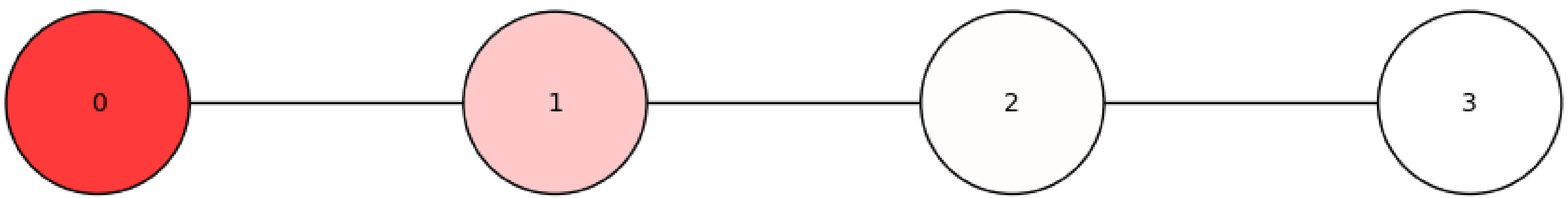
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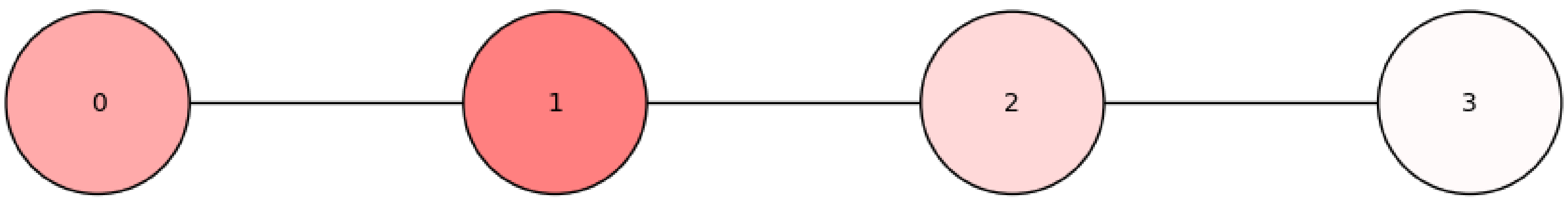
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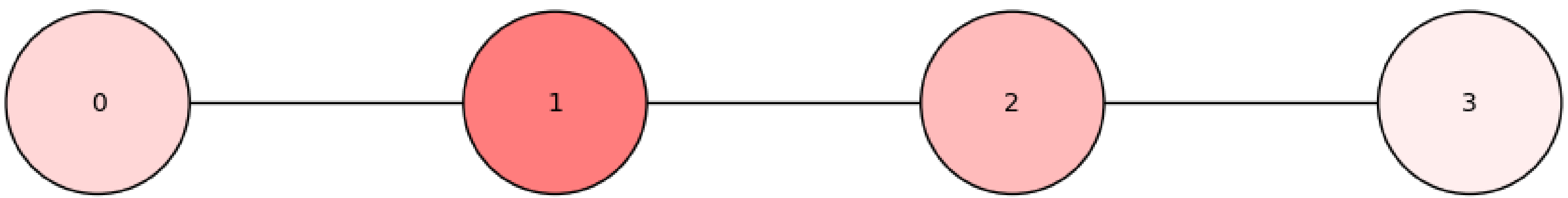
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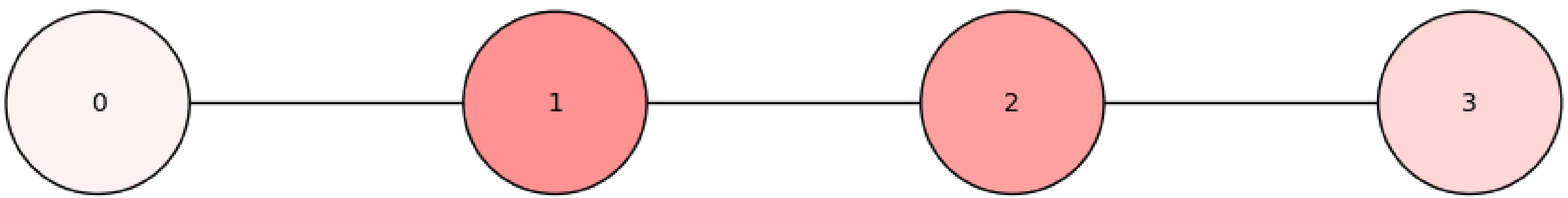
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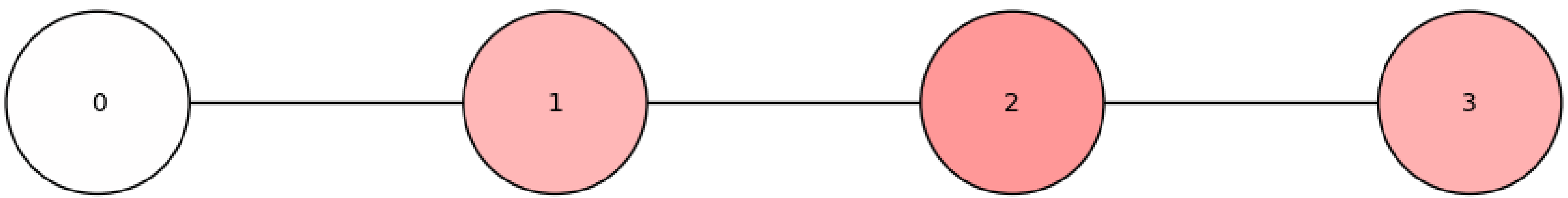
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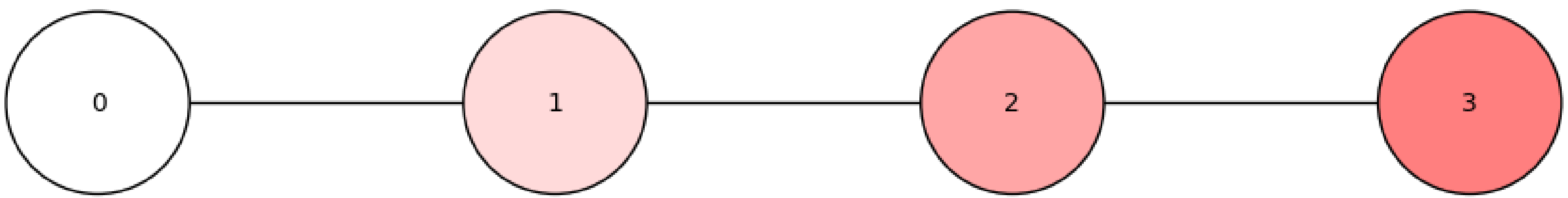
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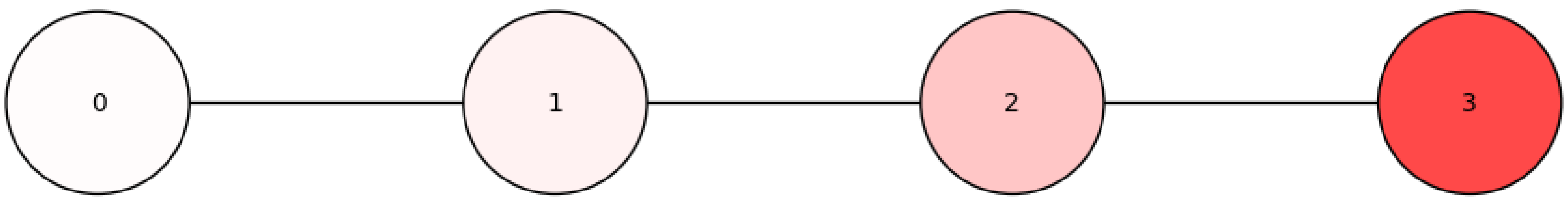
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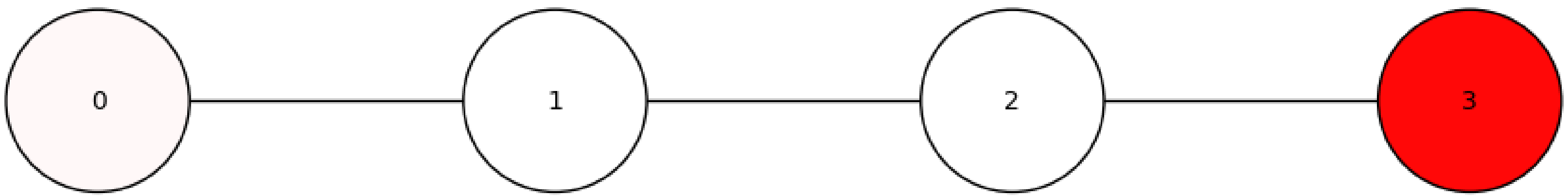
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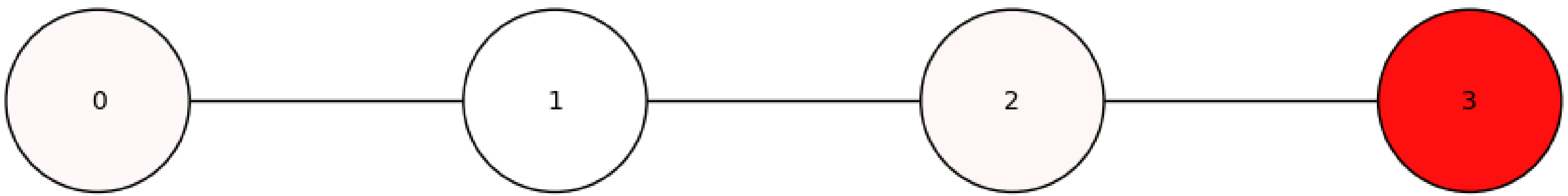
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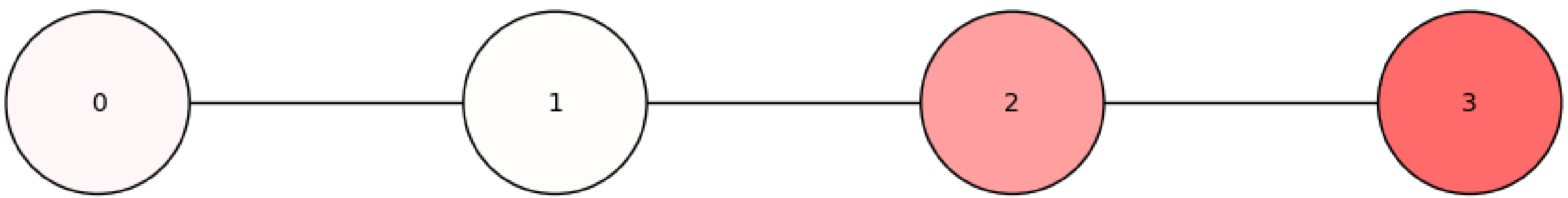
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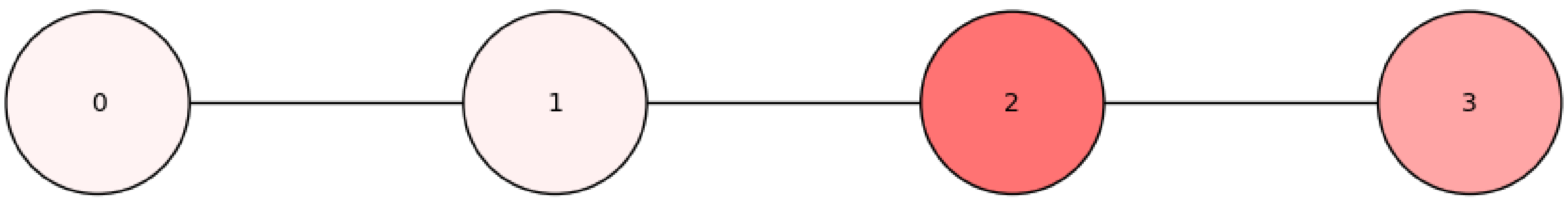
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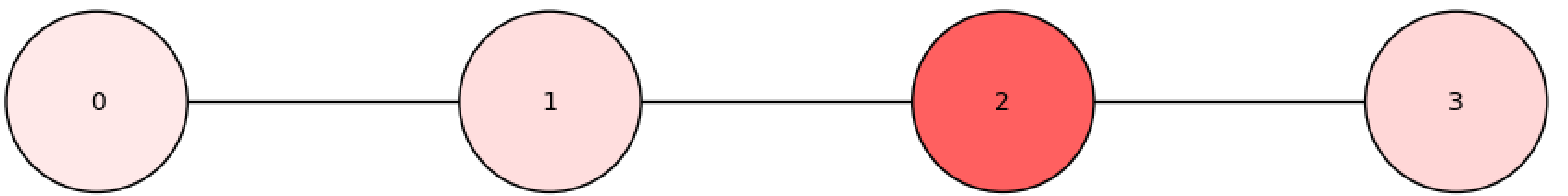
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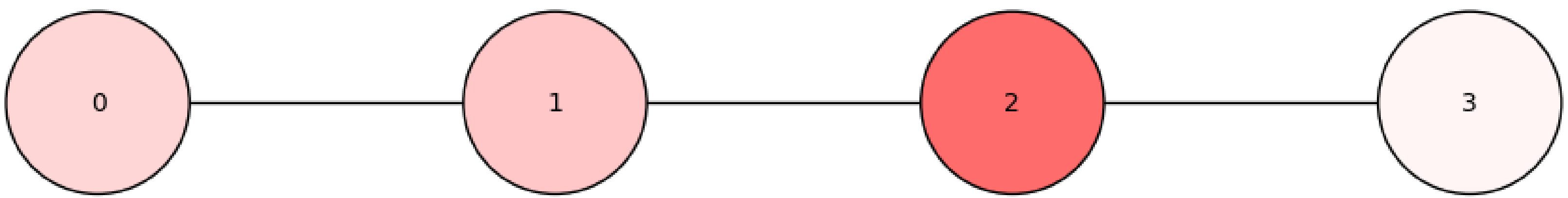
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Perfect state transfer: paths

Theorem (Godsil 2012)

The only paths which admit perfect state transfer are P_2 and P_3.

Perfect state transfer: paths

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Pretty good state transfer from a to b:

Perfect state transfer: paths

Theorem (Godsil 2012)
The only paths which admit perfect state transfer are P_2 and P_3.

Pretty good state transfer from a to b:

for every $\epsilon > 0$, there exists τ such that

$$||U(\tau)e_a - \gamma e_b|| < \epsilon$$

where $|\gamma| = 1$.

Pretty good state transfer

Theorem (Godsil, Kirkland, Severini, and Smith 2012)

P_n has pretty good state transfer
if and only if $n+1$ is a prime, twice a prime or a power of 2.

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P_n has pretty good state transfer between the ends if and only if $n+1$ is a prime, twice a prime or a power of 2.

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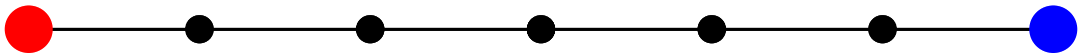
Theorem (Couthinho, Guo and van Bommel^2 2017)

P_n has pretty good state transfer between internal vxs if and only if $n+1 = 2^r p$ where p is a prime.

Pretty good state transfer



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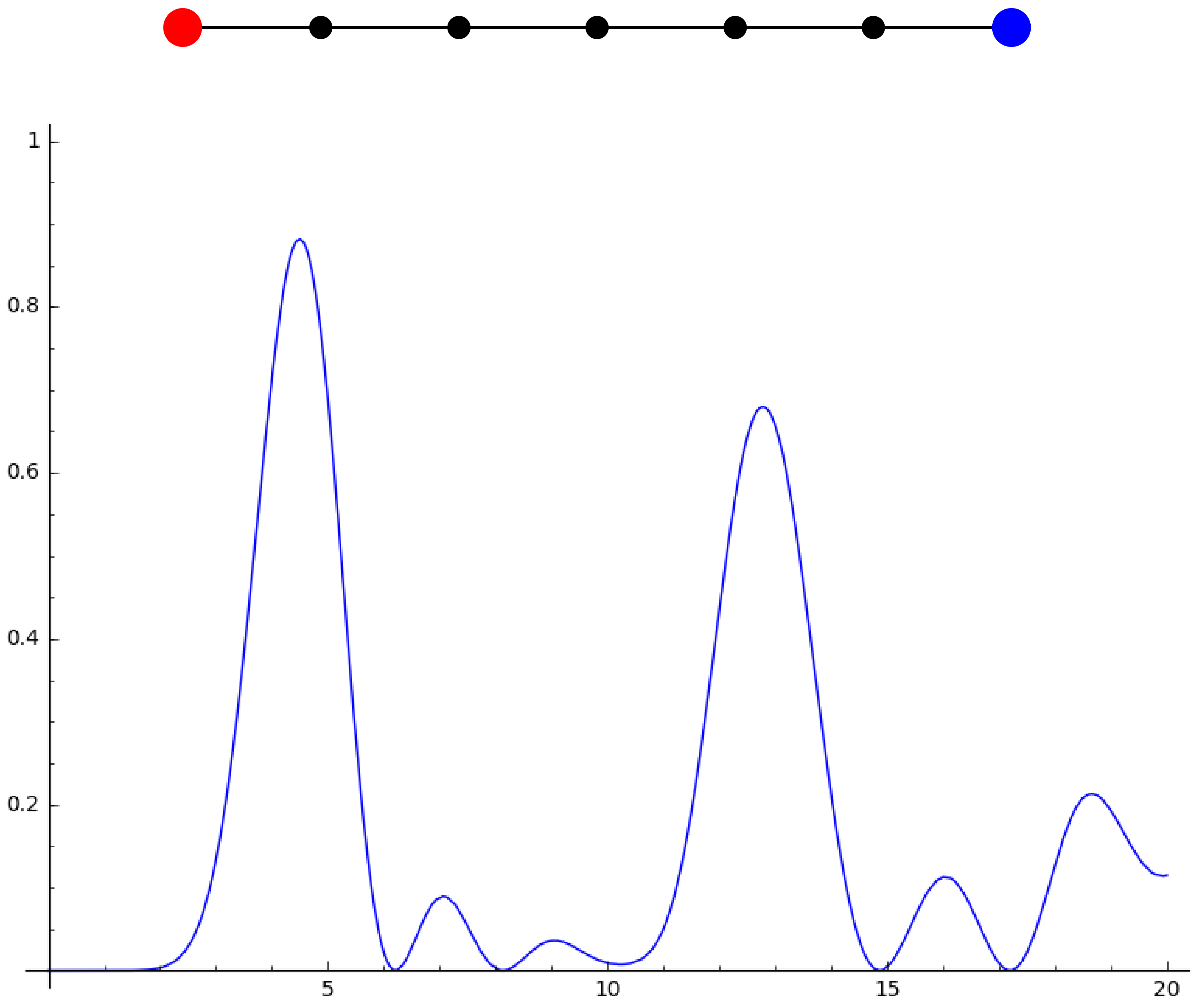

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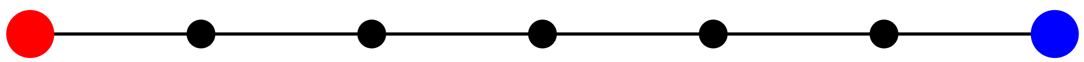

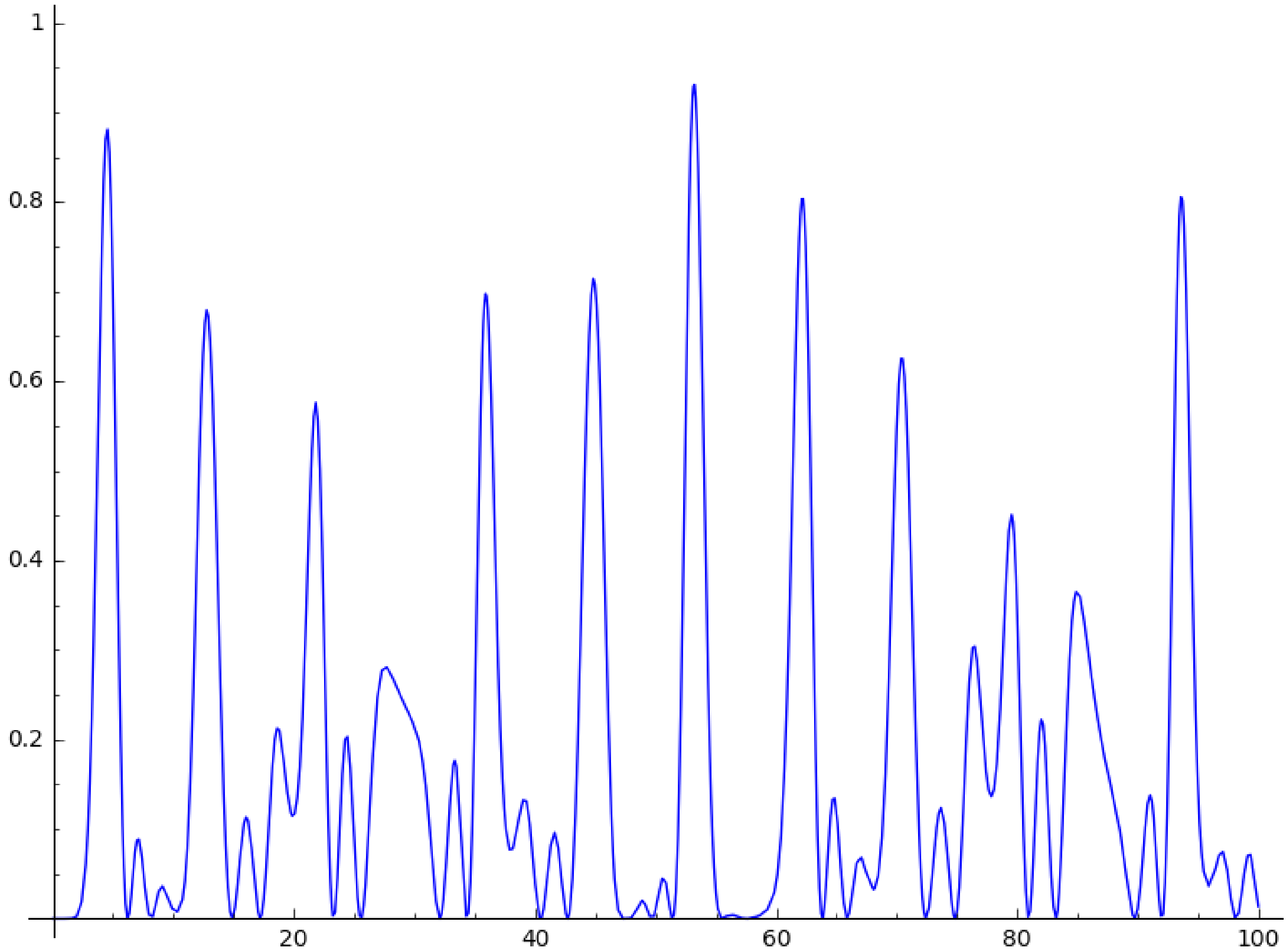

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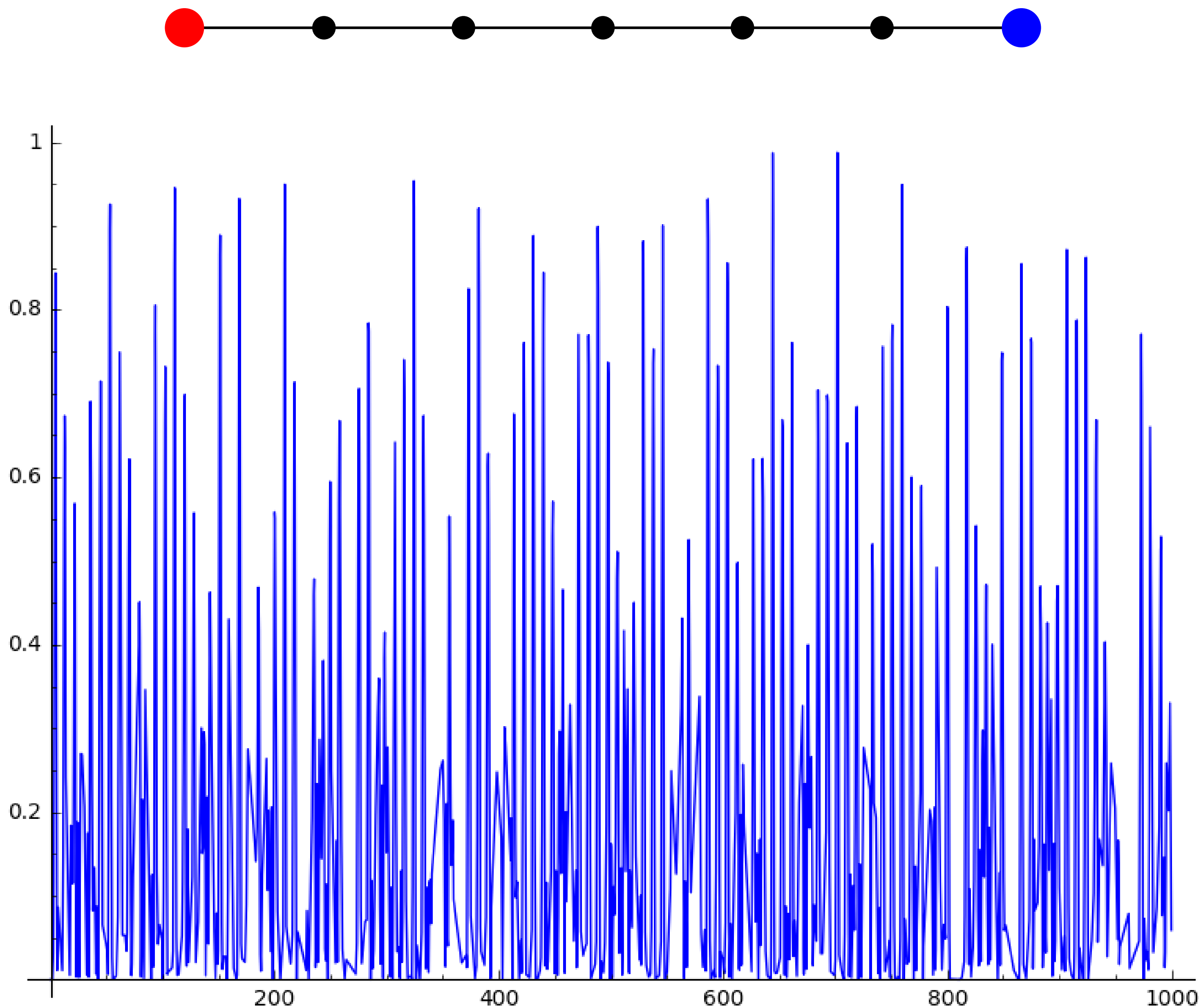

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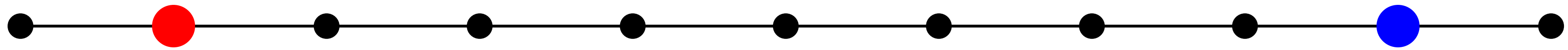
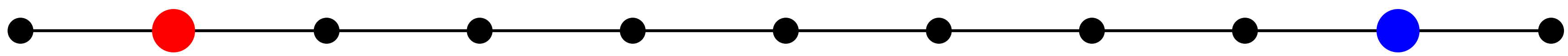
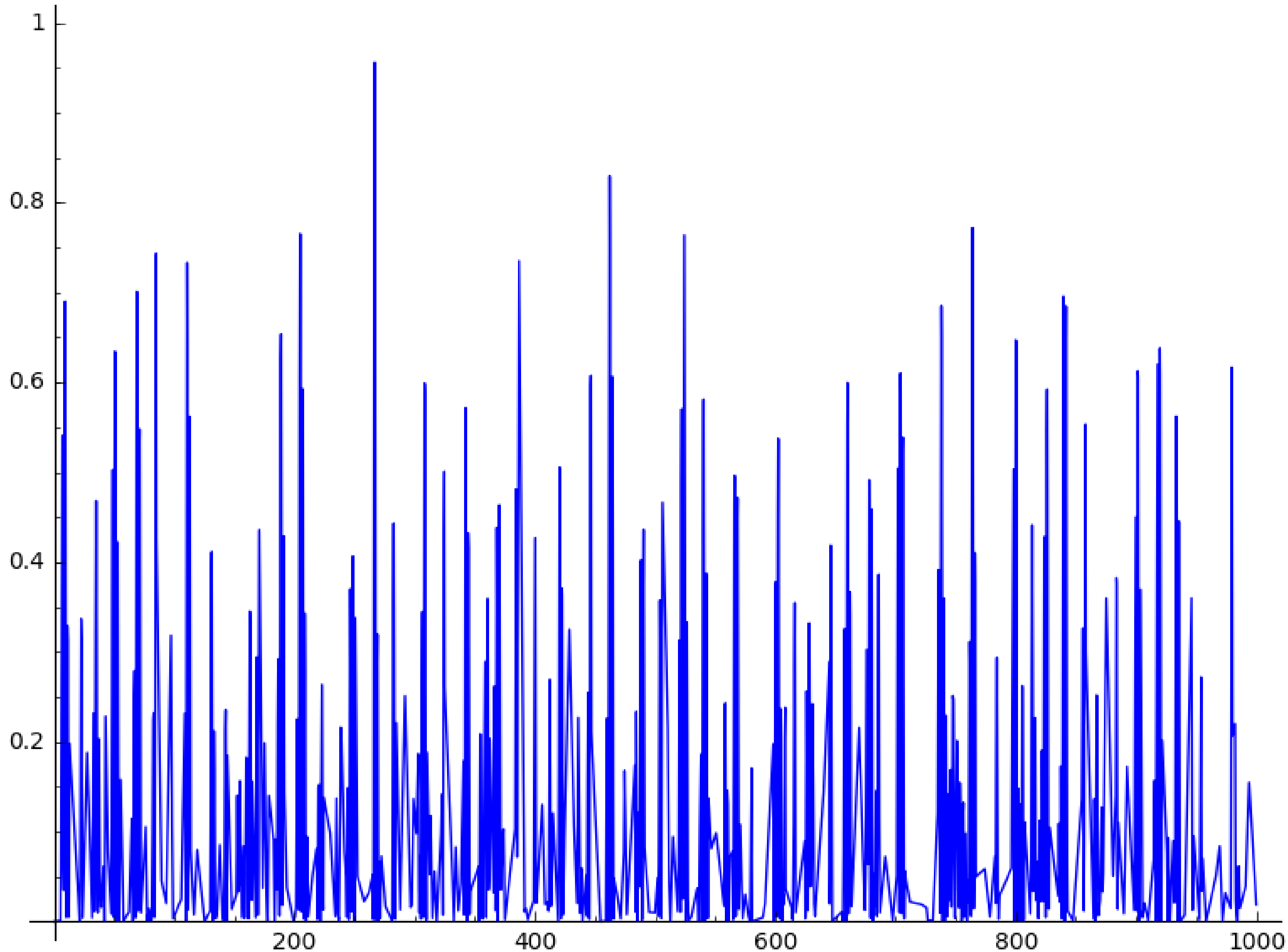
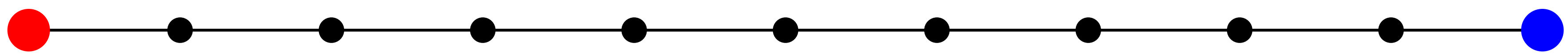


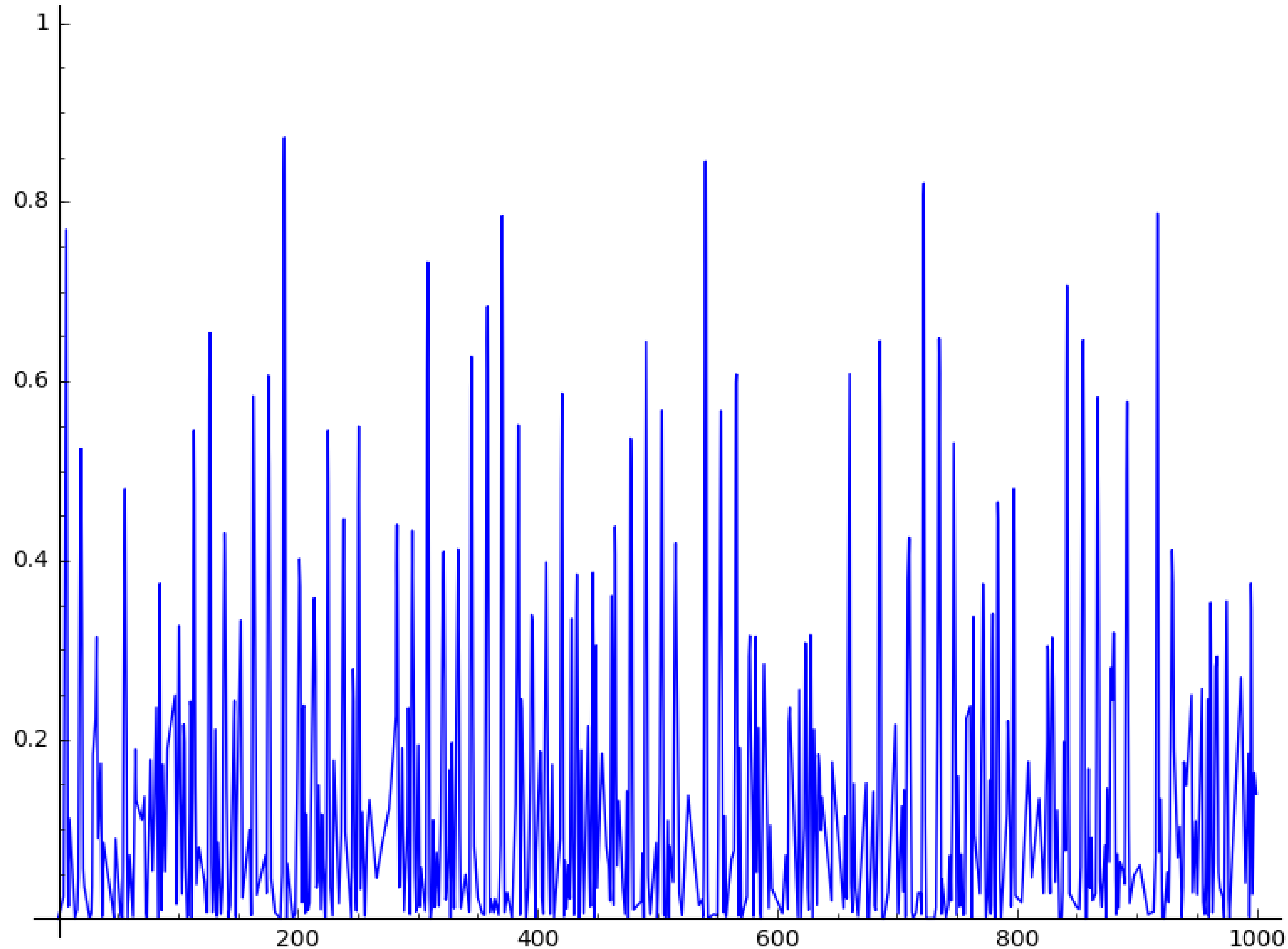
Pretty good state transfer



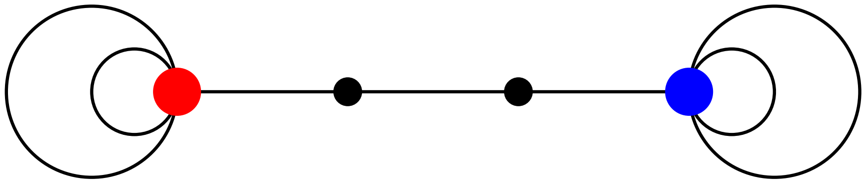


NO Pretty good state transfer





Perfect state transfer on paths revisited



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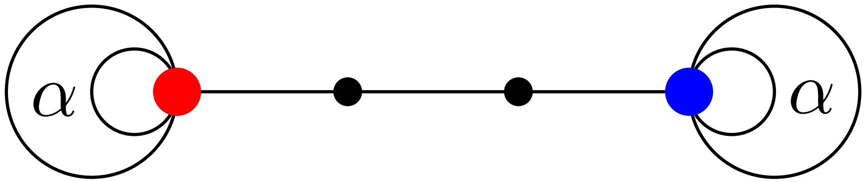

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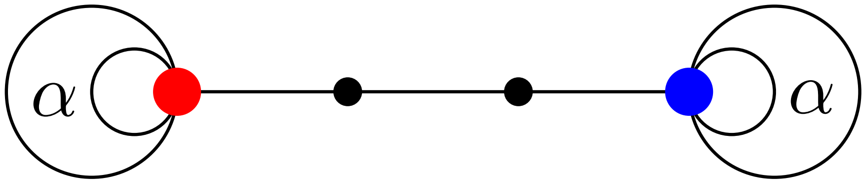

Conjecture (Casaccino, Lloyd, Mancini, and Severini '09)

For any n, one can find α so that there is perfect state transfer from  to  in P_n.

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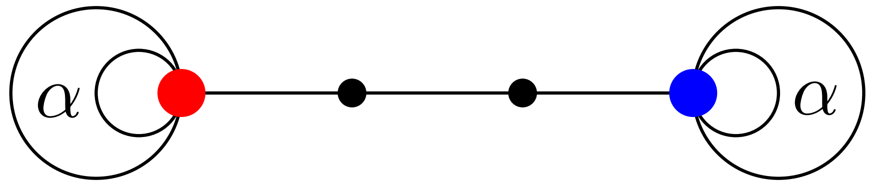

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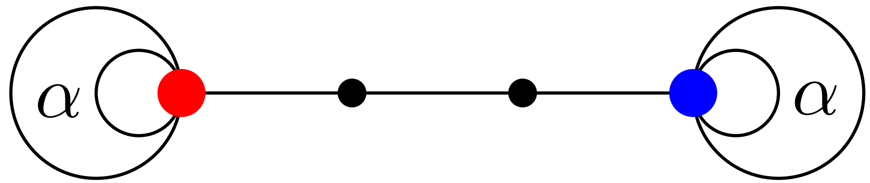

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Perfect state transfer in strongly regular graphs

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A strongly regular graph admits perfect state transfer if and only if it is the complement of a mK2 with m even.

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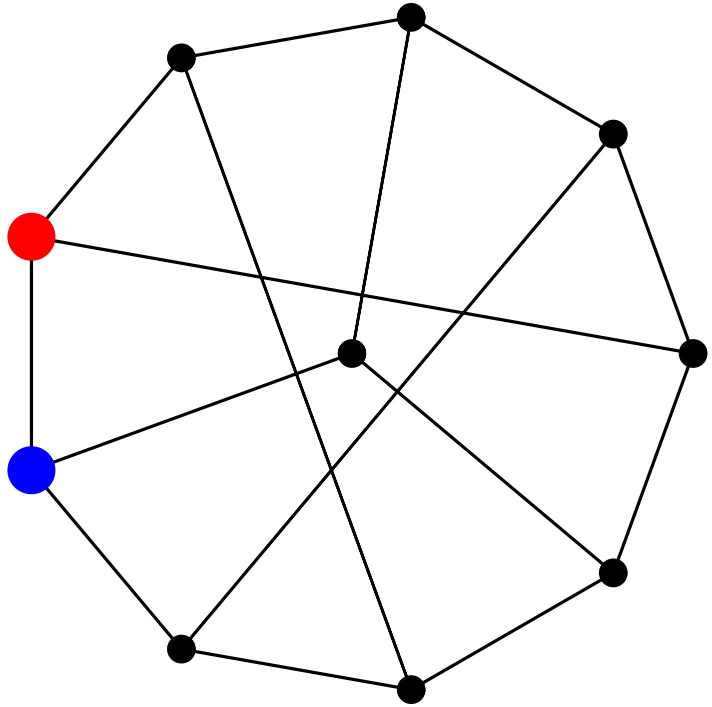

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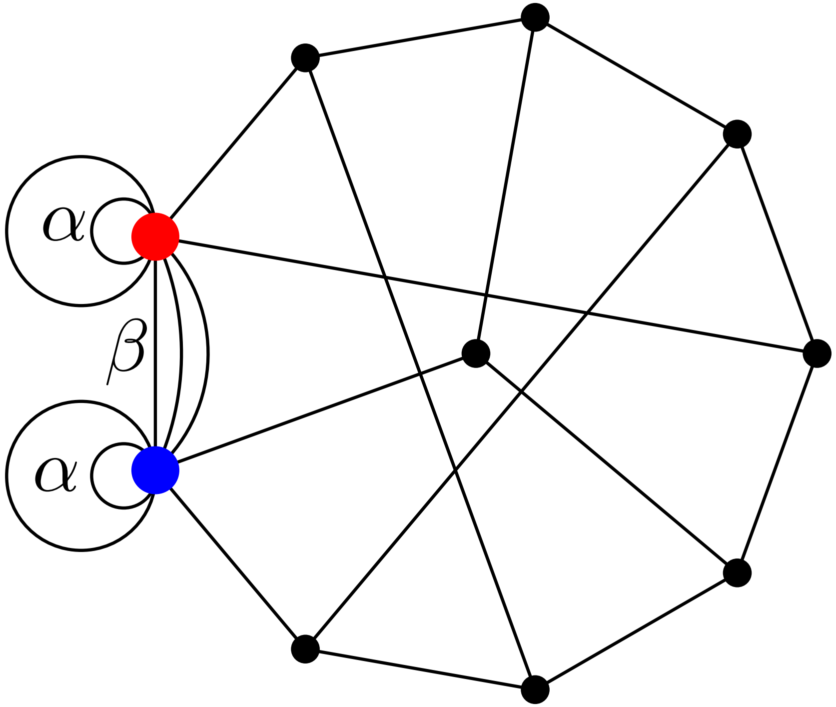

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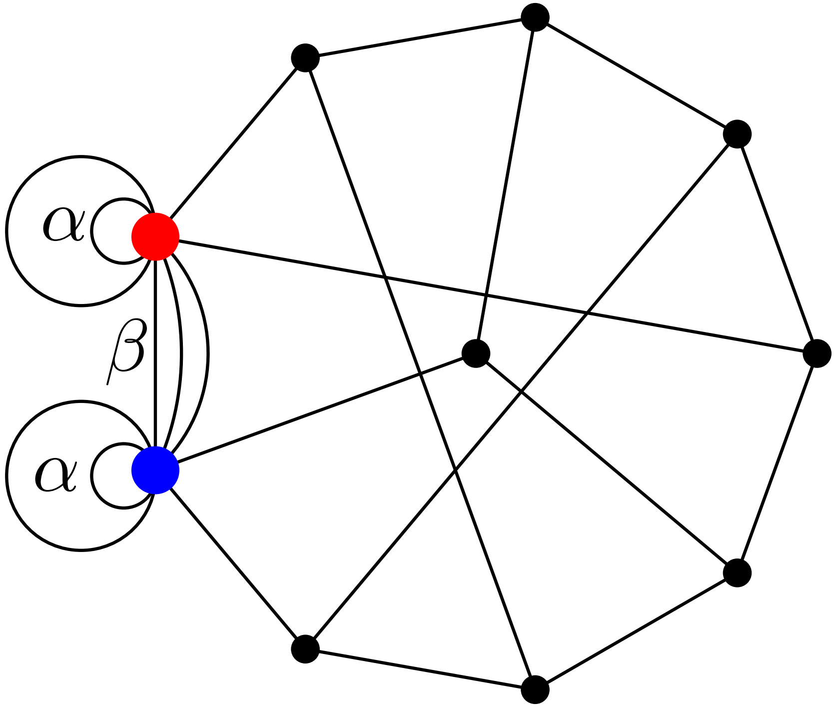

(α, β)-perturbation

Theorem (Godsil, Guo, Kempton and Lippner 2017+)

For any strongly regular graph coming from an orthogonal array, there exists α and β such that the (α, β)-perturbation admits perfect state transfer.

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- We determine the characteristic polynomial of the perturbed graph in terms of other characteristic polynomials, walk generating functions and α and β, for any graph.
- For an SRG, the perturbation has only 5 distinct eigenvalues and we are able to find sufficient conditions for p.s.t.

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For any strongly regular graph, there exists α and β such that the (α, β)-perturbation admits pretty good state transfer.

In fact, the "good" values of α, β are dense in the reals.

Open problems

Pretty good state transfer

Given an ϵ in a graph with pretty good state transfer, when does the "ϵ-close" state transfer occur?

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Perfect state transfer

Spectral characterization of graphs with a time τ such that

- uniform mixing at time τ; and
- perfect state transfer at time 2τ

Open problems

Pretty good state transfer

Given an ϵ in a graph with pretty good state transfer, when does the "ϵ-close" state transfer occur?

Perfect state transfer

Spectral characterization of graphs with a time τ such that

- uniform mixing at time τ; and
- periodicity at time 4τ ($U(4\tau)$ is a diagonal matrix)

Thanks!