

# The Smith group and the critical group of the Grassmann graph of lines in a finite projective space.

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AEGT 2017  
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In honor of Haemers, Lazebnik, and Woldar

August 9, 2017

This is joint work with Peter Sin.

# Outline

- 1 The skew-lines and Grassmann graphs
- 2 Integer invariants
- 3 Results
  - $p$ -part
  - $p'$ -part

- $\Gamma$ , graph with vertices the 2-dimensional subspaces of an  $n$ -dimensional vector space over  $\mathbb{F}_q$  ( $q = p^t$ ), adjacent when far apart
- $A$  adjacency matrix,  $L = D - A$  Laplacian matrix
- Complement graph denoted  $\Gamma'$
- $A', L'$
- Strongly regular with parameters

$$v' = \binom{n}{2}_q$$

$$k' = q(q+1) \binom{n-2}{1}_q$$

$$\lambda' = \binom{n-1}{1}_q + q^2 - 2$$

$$\mu' = (q+1)^2.$$

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- $|K(\Gamma)|$  counts number of spanning trees

## Elementary divisors

- $A: \mathbb{Z}_p^{V(\Gamma)} \rightarrow \mathbb{Z}_p^{V(\Gamma)}$
- $M_i = \left\{ x \in \mathbb{Z}_p^{V(\Gamma)} \mid Ax \in p^i \mathbb{Z}_p^{V(\Gamma)} \right\}$
- $N_i = \left\{ p^{-i} Ax \mid x \in M_i \right\}$
- Let  $e_i$  denote multiplicity of  $p^i$  as elementary divisor of  $A$
- 

$$\dim_{\mathbb{F}_p} \overline{M_i} = \dim_{\mathbb{F}_p} \overline{\ker(A)} + e_i + e_{i+1} + \cdots$$

and

$$\dim_{\mathbb{F}_p} \overline{N_i} = e_0 + e_1 + \cdots + e_i.$$

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- $A|_Y[A|_Y - qz_1 I] = q^3 z_2 I$
- $A$  has spectrum

$$q^4 \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_q, -q^2 \begin{bmatrix} n-3 \\ 1 \end{bmatrix}_q, q$$

with respective multiplicities  $1, \begin{bmatrix} n \\ 1 \end{bmatrix}_q - 1, \begin{bmatrix} n \\ 2 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ .

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$$\begin{bmatrix} n \\ 1 \end{bmatrix}_q - 1 \leq e_{2t} + \cdots + e_{3t} \tag{1}$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q \leq e_0 + \cdots + e_t. \tag{2}$$

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- In general, if  $A_{r,s}$  denotes the zero-intersection incidence matrix between  $r$ -spaces and  $s$ -spaces, then we have that

$$-A_{r,s} \equiv A_{r,1}A_{1,s} \pmod{p^t}$$

Skew lines,  $A$ 

## Theorem (Brouwer-D-Sin 2011)

Let  $e_i$  denote the multiplicity of  $p^i$  as a  $p$ -adic elementary divisor of  $A_{2,1}A_{1,2}$ .

- 1  $e_{4t} = 1$ .
- 2 For  $i \neq 4t$ ,

$$e_i = \sum_{\vec{s} \in \Gamma(i)} d(\vec{s}),$$

where

$$\Gamma(i) = \bigcup_{\substack{\alpha + \beta = i \\ 0 \leq \alpha \leq t \\ 0 \leq \beta \leq t}} \beta \mathcal{H} \cap \mathcal{H}_\alpha.$$

Summation over an empty set is interpreted to result in 0.

- $L$ : same result as for  $A$ , but no  $e_{4t}$ .
- $A'$ :  $p$ -part is cyclic of order  $p^t$
- $L'$ : no  $p$ -part

Notation: The vector space has dimension  $n$  over a field of  $q = p^t$  elements.

$(n, p, t)$	matrix	(elem. div. : multiplicity)
$(4, 2, 1)$	$A$	$(2 : 14), (2^2 : 8), (2^3 : 6), (2^4 : 1)$
	$L$	$(2 : 14), (2^2 : 8), (2^3 : 6)$ $(5 : 13)$ $(7 : 19)$
	$A'$	$(2 : 1)$ $(3 : 8), (3^2 : 14)$
	$L'$	$(3 : 8), (3^2 : 13)$ $(5 : 13)$ $(7 : 19)$

Notation: The vector space has dimension  $n$  over a field of  $q = p^t$  elements.

$(n, p, t)$	matrix	(elem. div. : multiplicity)
$(4, 2, 2)$	$A$	$(2 : 16), (2^2 : 220), (2^4 : 32), (2^5 : 16), (2^6 : 36), (2^8 : 1)$
	$L$	$(2 : 16), (2^2 : 220), (2^4 : 32), (2^5 : 16), (2^6 : 36)$ $(3 : 1), (3^2 : 271)$ $(7 : 271)$ $(17 : 83)$
	$A'$	$(2^2 : 1)$ $(3 : 84)$ $(5 : 190), (5^2 : 84)$
	$L'$	$(3 : 271)$ $(5 : 190), (5^2 : 83)$ $(7 : 271); (17 : 83)$



$p'$  part

- $\ell$ , a prime different than  $p$
- Structure of  $\mathbb{F}_\ell \text{GL}(n, q)$ -permutation module on 2-spaces is able to be understood. (few composition factors)
- We rely heavily on work of G. James
- $\mathbb{F}_\ell^{V(\Gamma)}$  has descending filtration with subquotients as Specht modules
- Straightforward arithmetic conditions determine the composition factors and multiplicities of the Specht modules

An example: Grassmann,  $L'$ 

Parameters:

$$\begin{aligned}k' &= q(q+1) \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q, & \mu' &= (q+1)^2, & \lambda' &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + q^2 - 2, \\r &= q(q+1) \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q - q^2 \begin{bmatrix} n-3 \\ 1 \end{bmatrix}_q + 1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \\s &= q(q+1) \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q + (q+1) = (q+1) \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q.\end{aligned}$$

$$\begin{aligned}|\mathcal{K}(\Gamma')| &= \frac{\begin{bmatrix} n \\ 1 \end{bmatrix}_q^f ((q+1) \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q)^g}{\begin{bmatrix} n \\ 2 \end{bmatrix}_q} \\ &= \begin{bmatrix} n \\ 1 \end{bmatrix}_q^{f-1} (q+1)^{g+1} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q^{g-1}.\end{aligned}$$

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- Case:  $\ell \nmid q+1$ ,  $\ell \mid \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ , and  $\ell \nmid \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$

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- $|K(\Gamma')| = \begin{bmatrix} n \\ 1 \end{bmatrix}_q^{f-1} (q+1)^{g+1} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q^{g-1}$ .
- Case:  $\ell \nmid q+1$ ,  $\ell \mid \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ , and  $\ell \nmid \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$
- Let  $a = v_\ell(\begin{bmatrix} n \\ 1 \end{bmatrix}_q)$ . Then  $v_\ell(|K(\Gamma')|) = a(f-1)$ .

An example: Grassmann,  $L'$ 

We have

$$f \leq \dim_F \overline{M}_a = 1 + \sum_{i \geq a} e_i,$$

Therefore,

$$a(f - 1) = v_\ell(|K(\Gamma')|) = \sum_{i \geq 0} i e_i \geq \sum_{i \geq a} i e_i \geq a \sum_{i \geq a} e_i \geq a(f - 1)$$

Since we must have equality throughout, it follows that  $e_i = 0$  unless  $i = 0$  or  $a$ , and that  $e_a = f - 1$ . Then as  $\sum_{i \geq 0} e_i = g + f$ , we have  $e_0 = g + 1$ .

Thank you for your attention!