Some results on the roots of the independence polynomial of graphs

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Algebraic and Extremal Graph Theory Conference

Aug 9, 2017

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Definition, notation

The independence polynomial of a graph G is

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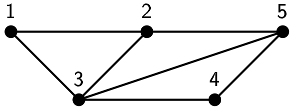

$$I(G,x) = 1 + 5x + 3x^2$$

Notation

For a generater function $f(x) = \sum_{k\geq 0} a_k x^k$, we will use the following notation

$$[x^k]f(x) = a_k$$

to denote the coefficient of x^k in $f(x)$.

Unimodality

A sequence $(b_k)_{k=0}^{n} \subset \mathbb{R}^+$ is

1. unimodal, if $\exists k \in \{0, \ldots n\}$, such that

$$b_0 \leq b_1 \leq \cdots \leq b_{k-1} \leq b_k \geq b_{k+1} \geq \cdots \geq b_n.$$

2. log-concave, if $\forall i \in \{1, \ldots, n-1\}$

$$b_i^2 \geq b_{i-1} b_{i+1}$$

Lemma (Newton)

If $p(x) = \sum_{i=0}^{n} b_i x^i$ has only real zeros, then the sequence $(b_k)_{k=0}^{n}$ is log-concave, therefore unimodal.

Unimodality on the independent subsets of the graphs

Question: Are the coefficients of $I(G,x)$ form an unimodal sequence, if

G is ...?	Answer:
connected	No
bipartite (Levit, Mandrescu)	No (Bhattacharyya, Kahn)
tree (Alavi et al.)	Open
line graph	Yes
claw-free graph	Yes

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Question(Galvin, Hilyard): Which trees have independence polynomial with only real zeros?

Stable-path tree

Let u be a fixed vertex of G, and choose a total ordering $\prec$ on $V(G)$.

Then the rooted tree $(T(G,u),\bar{u})$ defined as follows:

Let $N(u) = \{u_1 \prec \cdots \prec u_d\}$ and

$$G^i = G[V(G) \setminus \{u, u_1, u_2, \ldots, u_{i-1}\}]$$

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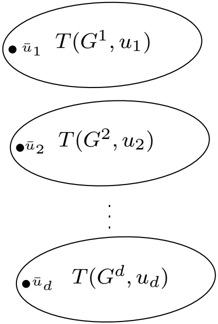

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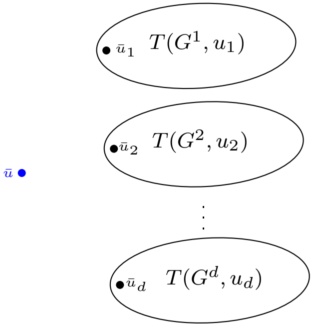

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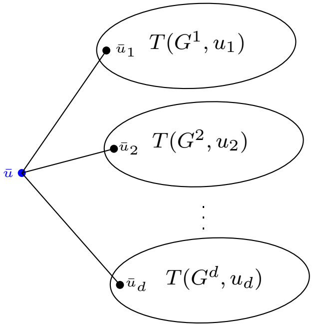

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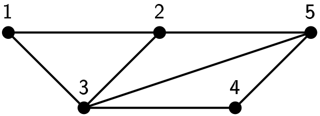

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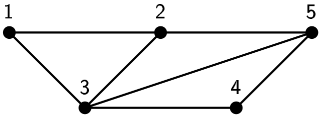

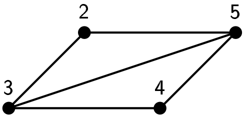

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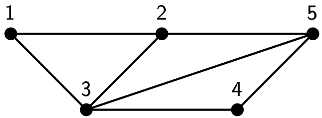

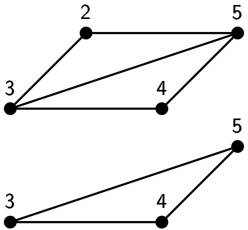

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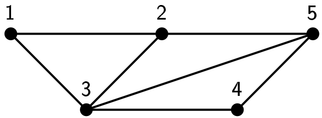

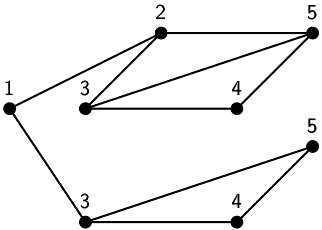

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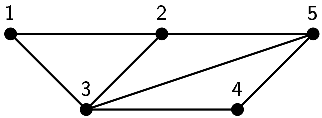
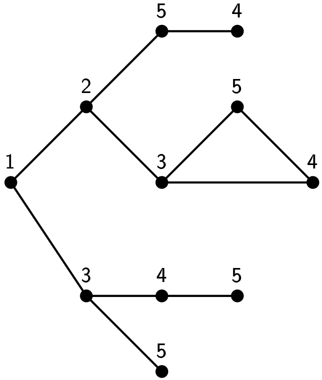
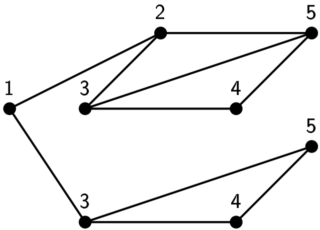
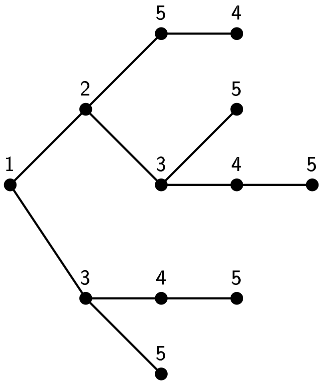






Example -cont.



Properties of the stable-path tree

Theorem (Scott, Sokal, Weitz)

Let G be a graph and $u \in V(G)$ be fixed. Then if $T = T(G, u)$, then

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2. *There exists a sequence of induced subgraphs $G_1,\dots,G_k$ of G, such that*

$$I(T,x) = I(G,x)I(G_1,x)\dots I(G_k,x).$$

Real-rooted independence polynomials

Theorem (Chudnovsky, Seymour)

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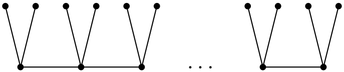
Theorem

Let G be a claw-free graph and $u \in V(G)$. Then $I(T(G,u),x)$ is real-rooted.
Moreover $I(G,x)$ divides $I(T(G,u),x)$.

Caterpillar H_n

The nth caterpillar (H_n) is the following tree on $3n$ vertices:



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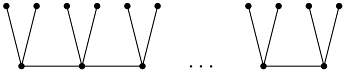

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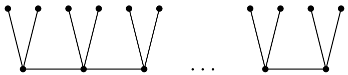

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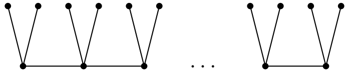

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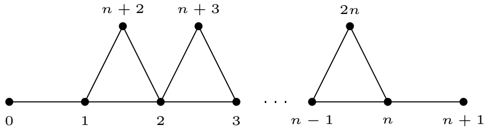

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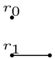

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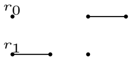

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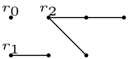

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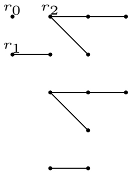

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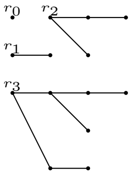

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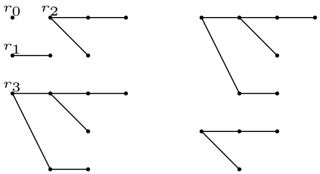

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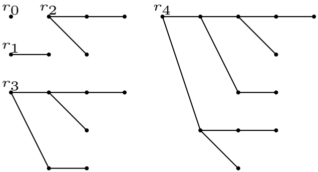

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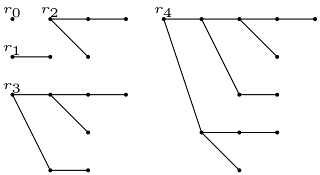

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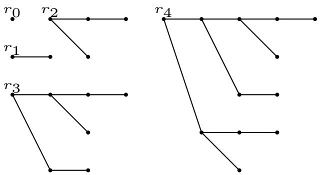

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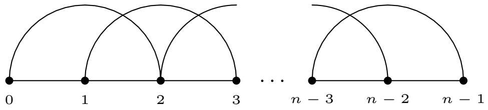
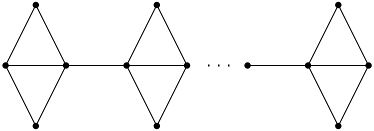
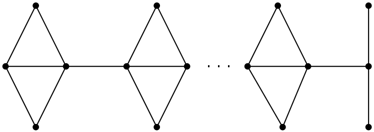


The M_n graph family



M_n when n is even.



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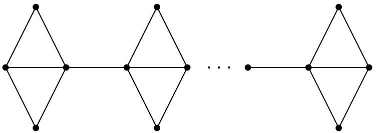

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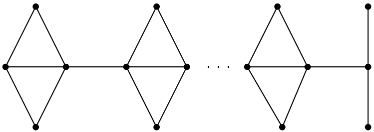

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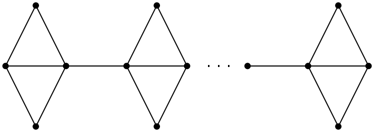

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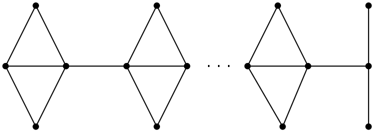

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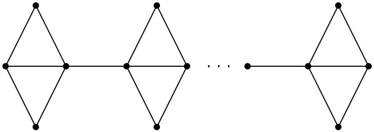

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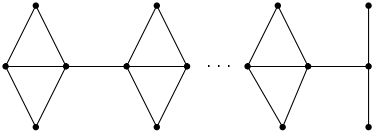

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As it turns out with a good ordering on the vertices the corresponding stable-path tree will be the nth caterpillar. So

$$I(M_n, x) \mid I(H_n, x),$$

and we already seen that $I(H_n, x)$ has only real zeros, therefore $I(M_n, x)$ has only real zeros.

Other real-rooted families

Trees:

Graphs:

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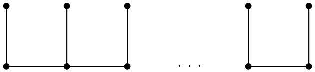

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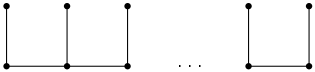

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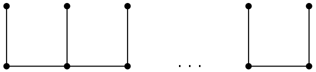

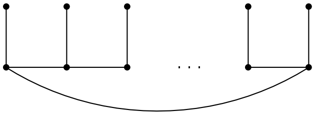

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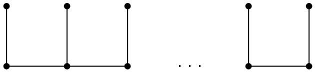

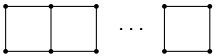

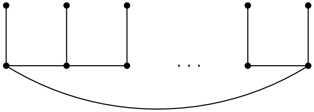

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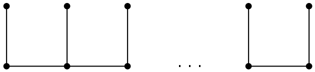
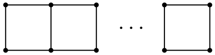
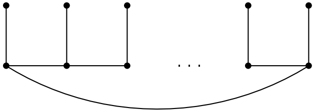
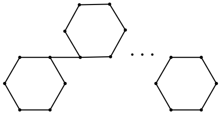
- Ladder graph (Zhu, Lu)
- Polyphenil ortho-chain (Alikhani, Jafari)









Are they the answer?

Is there a tree with real-rooted independence polynomial, such that it is not a stable-path tree of other then itself?

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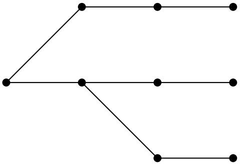
Yes. E.g.:



$$I(T, x) = (1+x)(1+8x+20x^2+16x^3+x^4)$$

Dictionary

Finite graphs	Infinite (rooted) graphs

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The main ingredient which enables us to move between the two “worlds”, is the “localization”. For any graph G, the coefficient

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depends only on the $k-1$-neighborhood of u in G, moreover it is a positive integer.

Infinite binary tree

Theorem

For the rooted binary tree (or 3-regular tree) there exists a measure μ on the real line, such that

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THANK YOU FOR YOUR ATTENTION!