On some cycles in linearized Wenger graphs

Ye Wang

Shanghai Lixin University of Accounting and Finance, Shanghai, 201209, China

AEGT, August 2017

Contents

Definition of Wenger graphs $W_m(q)$

- Let q be a prime power, and let $\mathbb{F}_q$ be the finite field of q elements. For any integer m with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m+1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by P_{m+1} and the other by L_{m+1}.

Definition of Wenger graphs $W_m(q)$

- Let q be a prime power, and let $\mathbb{F}_q$ be the finite field of q elements. For any integer m with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m+1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by P_{m+1} and the other by L_{m+1}.
- Elements of P_{m+1} will be called points and those of L_{m+1} lines.

Definition of Wenger graphs $W_m(q)$

- Let q be a prime power, and let $\mathbb{F}_q$ be the finite field of q elements. For any integer m with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m+1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by P_{m+1} and the other by L_{m+1}.
- Elements of P_{m+1} will be called points and those of L_{m+1} lines.
- The point $p = (p(1), p(2), \ldots, p(m+1))$ is adjacent to the line $l = [l(1), l(2), \ldots, l(m+1)]$ if and only if

$$p(i) + l(i) = p(1)(l(1))^{i-1},$$

for $i = 2, 3, \ldots, m+1$.

Definition of linearized Wenger graphs $L_m(q)$ (Cao et al. (2015))

- Let q be the power of prime p , and let $\mathbb{F}_q$ be the finite field of q elements. For any integer m with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m+1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by P_{m+1} and the other by L_{m+1}.

Definition of linearized Wenger graphs $L_m(q)$ (Cao et al. (2015))

- Let q be the power of prime p , and let $\mathbb{F}_q$ be the finite field of q elements. For any integer m with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m+1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by P_{m+1} and the other by L_{m+1}.
- Elements of P_{m+1} will be called points and those of L_{m+1} lines.

Definition of linearized Wenger graphs $L_m(q)$ (Cao et al. (2015))

- Let q be the power of prime p , and let $\mathbb{F}_q$ be the finite field of q elements. For any integer m with $m \geq 1$, the vertex set $V(W_m(q))$ is the disjoint union of two copies of the $m+1$ dimensional vector space $\mathbb{F}_q^{m+1}$ over finite field $\mathbb{F}_q$, one denoted by P_{m+1} and the other by L_{m+1}.
- Elements of P_{m+1} will be called points and those of L_{m+1} lines.
- The point $p = (p(1), p(2), \ldots, p(m+1))$ is adjacent to the line $l = [l(1), l(2), \ldots, l(m+1)]$ if and only if

$$p(i) + l(i) = p(1)(l(1))^{p^{i-2}},$$

for $i = 2, 3, \ldots, m+1$.

Property of linearized Wenger graphs $L_m(q)$

- The graphs $L_m(q)$ have $2q^{m+1}$ vertices, q^{m+2} edges and are q-regular.

Property of linearized Wenger graphs $L_m(q)$

- The graphs $L_m(q)$ have $2q^{m+1}$ vertices, q^{m+2} edges and are q-regular.
- Cao, Lu, Wan, Wang and Wang (2015) determine the girth, diameter and the spectrum of linearized Wenger graphs. For $q = p^e$, their results imply that the graphs $L_e(q)$ are expanders.

Property of linearized Wenger graphs $L_m(q)$

- The graphs $L_m(q)$ have $2q^{m+1}$ vertices, q^{m+2} edges and are q-regular.
- Cao, Lu, Wan, Wang and Wang (2015) determine the girth, diameter and the spectrum of linearized Wenger graphs. For $q = p^e$, their results imply that the graphs $L_e(q)$ are expanders.
- A work of Alexander, Lazebnik and Thomason (2016) implies that for a fixed e and large p, graphs $L_e(p^e)$ are hamiltonian.

Cycles in Wenger graphs

- For Wenger graphs $W_m(q)$, Shao, He and Shan (2008) showed that for any $m \geq 2$, and any k with $k \neq 5$, $4 \leq k \leq 2p$, $W_m(q)$ contains cycles of length $2k$.

Cycles in Wenger graphs

- For Wenger graphs $W_m(q)$, Shao, He and Shan (2008) showed that for any $m \geq 2$, and any k with $k \neq 5$, $4 \leq k \leq 2p$, $W_m(q)$ contains cycles of length $2k$.
- Wang, Lazebnik, Thomason (2014) extended their results by showing $W_m(q)$ contains cycles of length $2k$ for any $m \geq 2$ and any k with $k \neq 5$, $4 \leq k \leq 4p + 1$.

Cycles in linearized Wenger graphs

- What are the lengths of the cycles in linearized Wenger graphs?

Cycles in linearized Wenger graphs

- What are the lengths of the cycles in linearized Wenger graphs?

Theorem 1 (Wang (2017))

Let q be the power of prime p with $p \geq 3$. For any integer k with $3 \leq k \leq p^2$, $L_m(q)$ contains cycles of length $2k$.

The idea of constructing cycles in $L_m(q)$

Embedding cycles in partial planes to get even cycles of length from 6 to $2p^2-2p+2$ in $L_1(p)$.

Constructing cycle of length $2p^2$ in $L_1(p)$ and connecting some points and lines to get cycles of length from $2p^2-2p$ to $2p^2$.

→ Cycles of all even length from 6 to $2p^2$ in $L_1(p)$.

→ Cycles of all even length from 6 to $2p^2$ in $L_m(q)$.

Embedding cycles in partial planes

- Let p be a prime and $\mathbb{F}_p$ be the finite field of p elements. Let O be the point $(0,0)$ of a partial plane π which is constructed from projective plane $PG(2,p)$, and let $l_0, l_1, ..., l_{p-1}$ be the lines through point O in π.

Embedding cycles in partial planes

- Let p be a prime and $\mathbb{F}_p$ be the finite field of p elements. Let O be the point $(0,0)$ of a partial plane π which is constructed from projective plane $PG(2,p)$, and let $l_0, l_1, ..., l_{p-1}$ be the lines through point O in π.
- Here we take all the points on lines $l_0, l_1, \ldots, l_{p-1}$, and denote the point p by (x,y) with $x, y \in \mathbb{F}_p$ and the line l_k by $[k,0]$ with $k \in \mathbb{F}_p$. The point (x,y) is on the line l_k if and only if $y = kx$.

Embedding cycles in partial planes

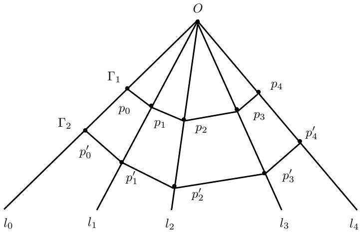
- Let p be a prime and $\mathbb{F}_p$ be the finite field of p elements. Let O be the point $(0,0)$ of a partial plane π which is constructed from projective plane $PG(2,p)$, and let $l_0, l_1, ..., l_{p-1}$ be the lines through point O in π.
- Here we take all the points on lines $l_0, l_1, \ldots, l_{p-1}$, and denote the point p by (x,y) with $x, y \in \mathbb{F}_p$ and the line l_k by $[k,0]$ with $k \in \mathbb{F}_p$. The point (x,y) is on the line l_k if and only if $y = kx$.
- We use $l_i + p$ to denote the line parallel to l_i that passes through p.

Embedding cycles in partial planes

- Let p be a prime and $\mathbb{F}_p$ be the finite field of p elements. Let O be the point $(0,0)$ of a partial plane π which is constructed from projective plane $PG(2,p)$, and let $l_0, l_1, ..., l_{p-1}$ be the lines through point O in π.
- Here we take all the points on lines $l_0, l_1, \ldots, l_{p-1}$, and denote the point p by (x,y) with $x, y \in \mathbb{F}_p$ and the line l_k by $[k,0]$ with $k \in \mathbb{F}_p$. The point (x,y) is on the line l_k if and only if $y = kx$.
- We use $l_i + p$ to denote the line parallel to l_i that passes through p.
- For prime p, we take a primitive element μ in $\mathbb{F}_p$ and let $\gamma = \frac{\mu}{\mu-1} \in \mathbb{F}_p$. Pick any point p_0 on l_0, different from O. Let p_{i+1} be the point of intersection of $l_{i+\gamma \pmod p} + p_i$ and $l_{i+1 \pmod p}$, for all $i = 0, 1, ..., p-2$.

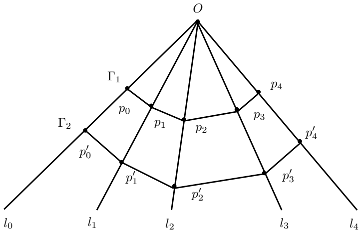
Example for $p = 5$

Figure 1: Two disjoint paths for $p = 5$



Example for $p = 5$

Figure 1: Two disjoint paths for $p = 5$



Lemma 2

Let $p_0 \neq p_0' \in l_0$ and let Γ_1, Γ_2 be two distinct paths with $p_0 \in \Gamma_1, p_0' \in \Gamma_2$, then Γ_1, Γ_2 share neither points nor lines.

Lemma 3

Let p be an odd prime. For any integer k with $3 \leq k \leq p^2 - p + 1$, cycles of length k can be embedded in π.

Lemma 3

Let p be an odd prime. For any integer k with $3 \leq k \leq p^2 - p + 1$, cycles of length k can be embedded in π.

Lemma 4

For odd prime p, $L_1(p)$ contains cycles of length 2k with $3 \leq k \leq p^2 - p + 1$.

Notations

- Let P_2 and L_2 denote the point set and line set in $L_1(p)$, respectively. And let $p_1 l_1 p_2 l_2 \dots p_{p^2} l_{p^2}$ be a walk of length $2p^2$ in $L_1(p)$ with $p_i \in P_2$ and $l_i \in L_2$, $1 \leq i \leq p^2$.

Notations

- Let P_2 and L_2 denote the point set and line set in $L_1(p)$, respectively. And let $p_1 l_1 p_2 l_2 \dots p_{p^2} l_{p^2}$ be a walk of length $2p^2$ in $L_1(p)$ with $p_i \in P_2$ and $l_i \in L_2$, $1 \le i \le p^2$.
- For $p_i \in P_2$ and $l_i \in L_2$ with $1 \le i \le p^2$ and $1 \le j \le m+1$, denote $p_i(j)$ the jth component of point p_i and $l_i(j)$ the jth component of line l_i.

- We take the first components of p_i and l_i in table form.

- We take the first components of p_i and l_i in table form.

i	1	2	...	p	$p+1$	$p+2$	...	$2p$	...	p^2
$p_i(1)$	1	2	...	0	2	3	...	1	...	$p-1$
$l_i(1)$	1	2	...	0	2	3	...	1	...	$p-1$

Table 1: The first components of p_i and l_i

- We take the first components of p_i and l_i in table form.

i	1	2	...	p	$p+1$	$p+2$	...	$2p$	...	p^2
$p_i(1)$	1	2	...	0	2	3	...	1	...	$p-1$
$l_i(1)$	1	2	...	0	2	3	...	1	...	$p-1$

Table 1: The first components of p_i and l_i

- $p_1 l_1 p_2 l_2 \ldots p_{p^2} l_{p^2}$ is a cycle of length $2p^2$.

- We take the first components of p_i and l_i in table form.

i	1	2	...	p	$p+1$	$p+2$	...	$2p$	...	p^2
$p_i(1)$	1	2	...	0	2	3	...	1	...	$p-1$
$l_i(1)$	1	2	...	0	2	3	...	1	...	$p-1$

Table 1: The first components of p_i and l_i

- $p_1 l_1 p_2 l_2 \dots p_{p^2} l_{p^2}$ is a cycle of length $2p^2$.

Lemma 5

For odd prime p, $L_1(p)$ is Hamiltonian.

Connecting points and lines in C_{2p^2}

- l_{ip-2} is adjacent to $p_{(i+1)p-4}$, and l_{ip-1} is adjacent to $p_{(i+1)p-3}$, for $i = 1, 2, \ldots, p-1$.
- l_{p^2-2} is adjacent to p_{p-4}, and l_{p^2-1} is adjacent to p_{p-3}.

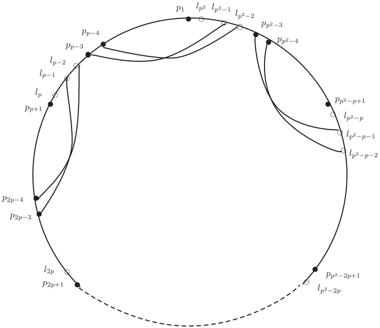


Figure 2: cycles in $L_1(p)$

All even cycles in $L_1(p)$

1. For odd prime p, $L_1(p)$ contains cycles of length $2k$, where $3 \leq k \leq p^2 - p + 1$.

All even cycles in $L_1(p)$

1. For odd prime p, $L_1(p)$ contains cycles of length $2k$, where $3 \leq k \leq p^2 - p + 1$.
2. For odd prime p, $L_1(p)$ contains cycles of length $2k$, where $p^2 - p \leq k \leq p^2$.

All even cycles in $L_1(p)$

1. For odd prime p, $L_1(p)$ contains cycles of length $2k$, where $3 \leq k \leq p^2 - p + 1$.
2. For odd prime p, $L_1(p)$ contains cycles of length $2k$, where $p^2 - p \leq k \leq p^2$.

Lemma 6

For odd prime p and any integer k with $3 \leq k \leq p^2$, $L_1(p)$ contains cycles C_{2k}.

Cycles in $L_m(q)$

Lemma 7

For $m \geq 1$ and odd prime p, $L_m(p)$ consists of p^{m-1} components each isomorphic to $L_1(p)$.

Cycles in $L_m(q)$

Lemma 7

For $m \geq 1$ and odd prime p, $L_m(p)$ consists of p^{m-1} components each isomorphic to $L_1(p)$.

Theorem 8

Let q be the power of prime p with $p \geq 3$. For any integer k with $3 \leq k \leq p^2$, $L_m(q)$ contains cycles of length $2k$.

Future work

- Find more cycles in Wenger graphs and linearized Wenger graphs.

Future work

- Find more cycles in Wenger graphs and linearized Wenger graphs.

Conjecture 1

For every $n \geq 1$*, and every prime power* q*,* $q \geq 3$*,* $W_n(q)$ *contains cycles of length* $2k$*, where* $4 \leq k \leq q^{n+1}$ *and* $k \neq 5$*.*

Thank you!