An extremal problem involving 4-cycles and planar polynomials

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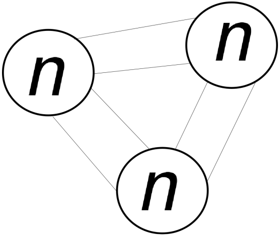
Work supported by the Simons Foundation

Algebraic and Extremal Graph Theory

Introduction

Suppose G is a 3-partite graph with n vertices in each part.



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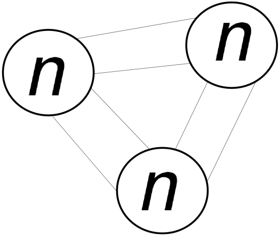


Question: How many triangles can appear in G?

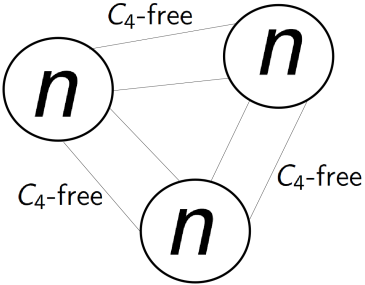
Answer: If there are no further assumptions on G, then we can have

$$n^3 \text{ triangles.}$$

Introduction

Let us assume that

G has no 4-cycle between any two parts.



This question was asked by Fischer and Matoušek (2001).

Introduction

Probabilistic: A standard probabilistic argument shows that there is such a G with about n triangles.

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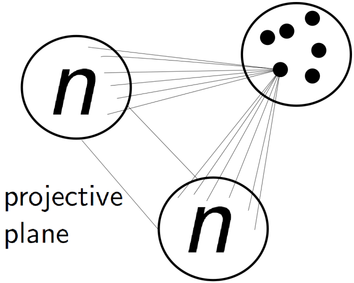
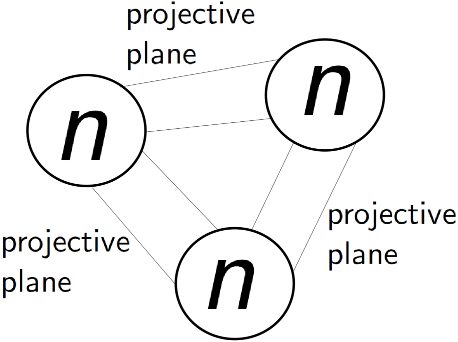


This graph will have about $n^{3/2}$ triangles.

Algebraic + Probabilistic:



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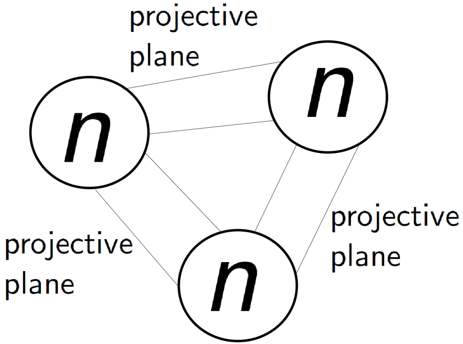

Putting projective planes at random between the parts gives a lower bound of

$$n^3 \left(\frac{1}{\sqrt{n}}\right)^3 = n^{3/2} \text{ triangles.}$$

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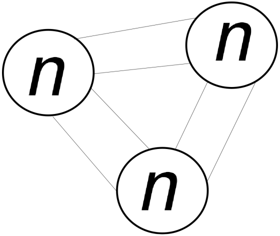
Upper bound: Suppose G has parts A, B, and C.

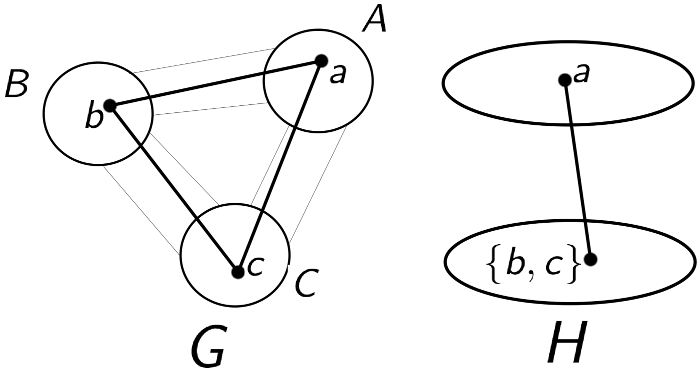


Introduction

Let H be the bipartite graph with parts A and $E(B, C)$ where

$$a \in A \text{ is adjacent to } \{b, c\} \in E(B, C)$$

if and only if a, b, c is a triangle in G.



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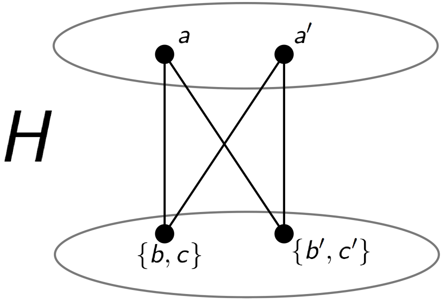
- The graph H does not contain a C_4.



If $b \neq b'$, then $aba'b'$ is a C_4 in G between A and B.

Introduction

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so that

$$\begin{aligned} \#\text{ of triangles in G} = e(H) &\lesssim |A| e(B, C)^{1/2} \\ &\lesssim |A|(|B||C|^{1/2})^{1/2} \\ &= n^{7/4} \end{aligned}$$

*The second $\lesssim$ is because there is no C_4 between B and C.

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for the maximum number of triangles in a 3-partite graph with n vertices in each part, and no C_4 between two parts.

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$$(1 - o(1))n^{3/2} \leq \triangle(n) \leq (1 + o(1))n^{7/4}.$$

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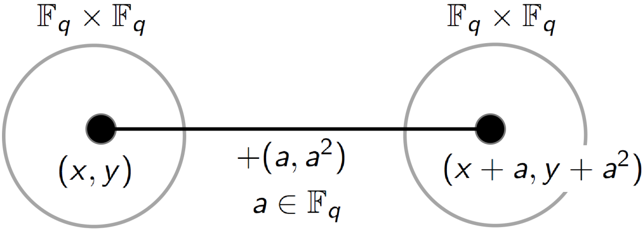
Goal: Improve the lower bound.

A First Attempt

A simple construction is as follows (q is a power of an odd prime):



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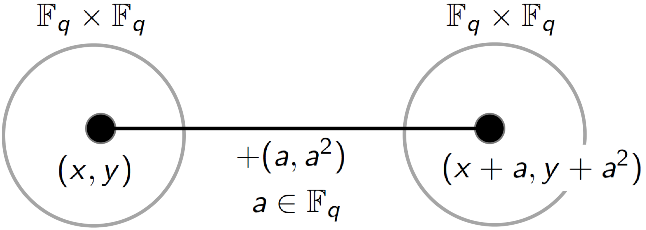
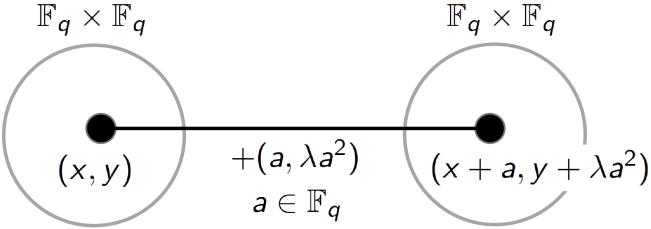


Introduce a parameter $\lambda \in \mathbb{F}_q \backslash \{0\}$.



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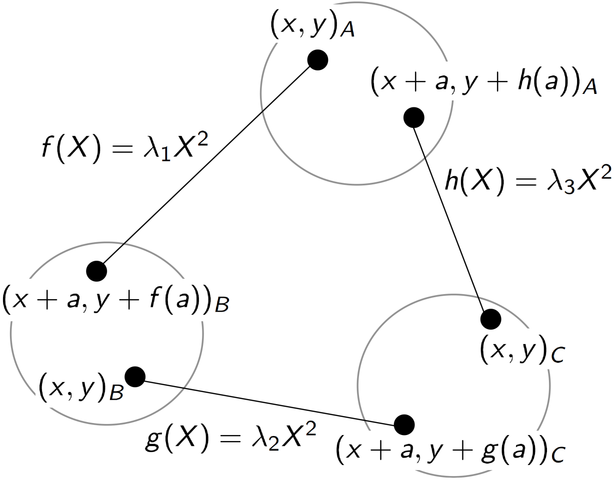
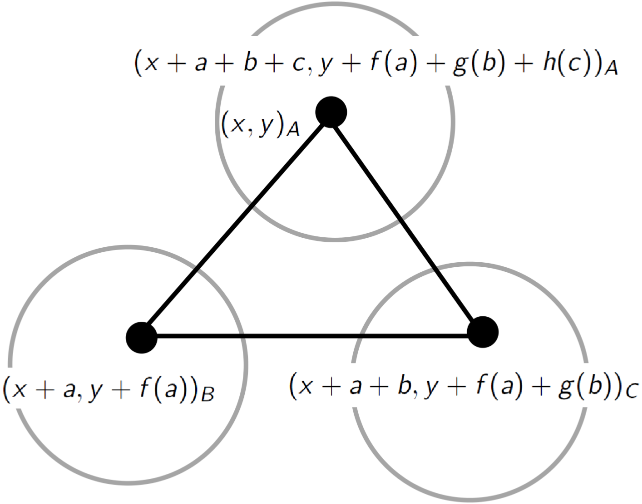


A First Attempt

Question: How many triangles?



A First Attempt

Key Idea: To get many triangles, we need many solutions to

$$0 = a + b + c \qquad 0 = f(a) + g(b) + h(c)$$

or equivalently,

$$0 = a + b + c \qquad 0 = \lambda_1 a^2 + \lambda_2 b^2 + \lambda_3 c^2.$$

A First Attempt

Assume $\lambda_3 = 1$ and use $c = -a - b$ with $0 = f(a) + g(b) + h(c)$ to get

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This shows that $\frac{a}{b}$ is a root of

$$p_{\lambda_1,\lambda_2}(X) = (\lambda_1 + 1)X^2 + 2X + (\lambda_2 + 1).$$

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  $$\frac{a}{b} = \zeta_i$$

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- We choose $(x, y)_A$ to get our triangle

$$(x, y)_A,\ (x + a, y + \lambda_1 a^2)_B,\ (x + a + b, y + \lambda_1 a^2 + \lambda_2 b^2)_C.$$

Altogether, this gives $2(q-1)q^2$ triangles and shows

$$(2 - o(1))n^{3/2} \leq \triangle(n)$$

which improves the previous bound by a factor of 2.

A First Attempt

Limitation: The polynomial

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Solution: Make $p_{\lambda_1,\lambda_2}(X)$ have higher degree.

<u>Our graph must have no C_4 between two parts</u>.

A Second Attempt

Let q be a power of an odd prime.

A polynomial $f(x) \in \mathbb{F}_q[X]$ is a **planar polynomial** if for each $a \in \mathbb{F}_q \backslash \{0\}$, the map $\phi_a : \mathbb{F}_q \to \mathbb{F}_q$ defined by

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Planar polynomials were first defined by Dembowski and Ostrom in 1968.

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Example: The quadratic $f(X) = \lambda_1 X^2$ is a planar polynomial: if $a \neq 0$, then

$$\phi_a(x) = f(x+a) - f(x) = \lambda_1(x+a)^2 - \lambda_1 x^2 = \lambda_1(2xa + a^2)$$

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3. $f(X) = X^{\frac{1}{2}(3^k+1)}$ over $\mathbb{F}_{3^e}$ for k odd and $\gcd(k,e) = 1$.

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The last two have degrees that can be made arbitrarily large:

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Recall that we want

$$0 = a + b + c \quad \text{and} \quad f(a) + g(b) + h(c) = 0$$

and in quadratic case, we ended up with $\frac{a}{b}$ being a root of

$$p_{\lambda_1,\lambda_2}(X) = (\lambda_1 + 1)X^2 + 2X + (\lambda_2 + 1).$$

A Second Attempt

Using

$$X^{q+1}$$

which is planar over $\mathbb{F}_{q^3}$ whenever q is a power of an odd prime, we get that $\frac{a}{b}$ is a root of

$$w_{\lambda_1,\lambda_2}(X) = (\lambda_1 + 1)X^{q+1} + X^q + X + (\lambda_2 + 1).$$

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This polynomial has degree $q+1$ if $\lambda_1 \neq -1$, and so we have a chance at choosing λ_1 and λ_2 so that we get much more than just 2 roots.

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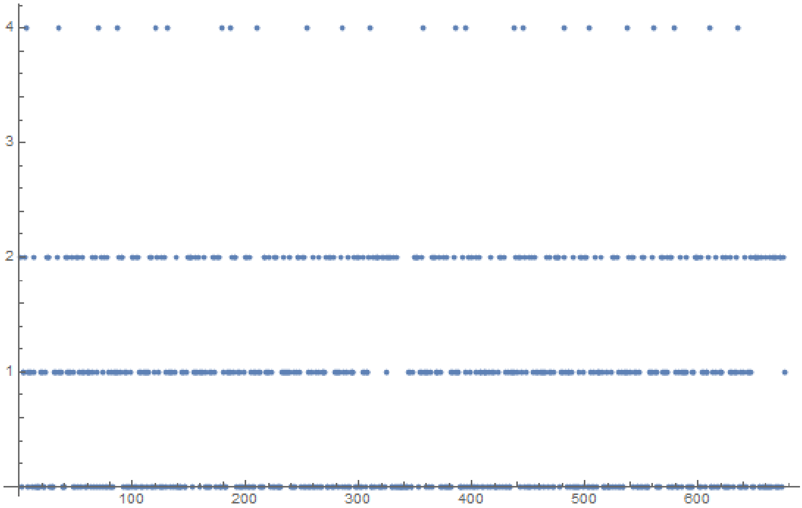

Counting roots over all $(3^3 - 1)^2$ choices of λ_1, λ_2 in $w_{\lambda_1,\lambda_2}(X)$.

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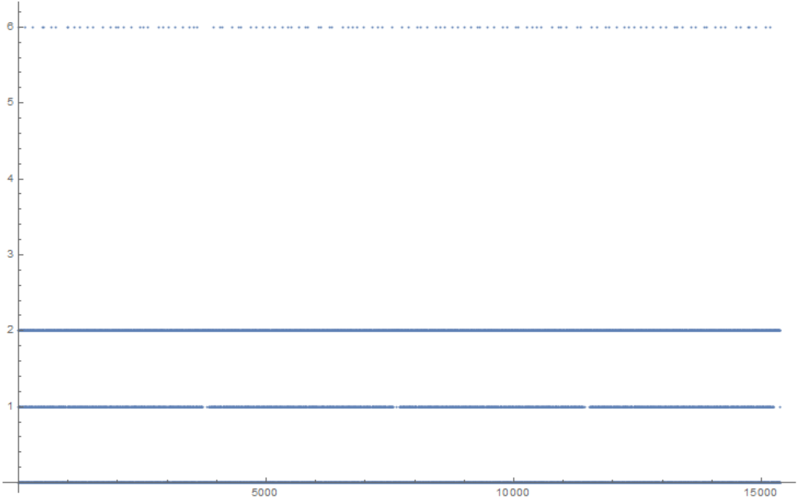

Counting roots over all $(5^3 - 1)^2$ choices of λ_1, λ_2 in $w_{\lambda_1,\lambda_2}(X)$.

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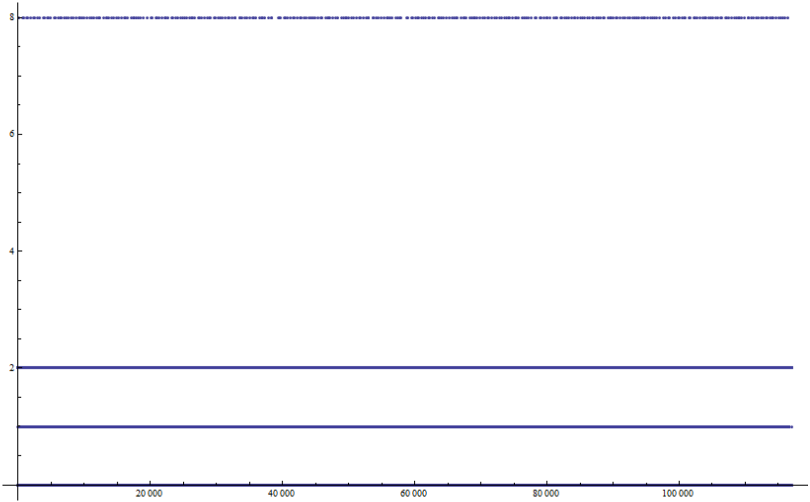

Counting roots over all $(7^3 - 1)^2$ choices of λ_1, λ_2 in $w_{\lambda_1,\lambda_2}(X)$.

Main Result

Theorem (Coulter, Matthews, T, 2017)

Let q be a power of an odd prime and $a \in \mathbb{F}_{q^3} \backslash \{0\}$.

The polynomial

$$g_a(X) = X^{q+1} + a^{-1}(X^q + X) + a^{-1-q} + a^{-1-q^2} - a^{-q^2-q}$$

will have $q+1$ distinct roots in $\mathbb{F}_{q^3}$ whenever $a \in \mathbb{F}_{q^3} \backslash \mathbb{F}_q$.

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This is equivalent to the statement that

$$f_a(X) = X^{q+1} + a^{-1}(X^q + X) + \mathrm{N}(a^{-1})(\mathrm{Tr}(a) - 2a)$$

has $q+1$ distinct roots in $\mathbb{F}_{q^3}$ whenever $a \in \mathbb{F}_{q^3} \backslash \mathbb{F}_q$.

This result tells us how we should choose λ_1, λ_2 so that $w_{\lambda_1,\lambda_2}(X)$ has many roots.

Main Result

- If ζ_i is root, we choose $b \in \mathbb{F}_{q^3} \backslash \{0\}$ and let

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Altogether, this gives $(q+1)(q^3-1)q^6$ triangles (here $n = q^6$) and shows

$$(1 - o(1))n^{5/3} \leq \triangle(n)$$

which improves the previous bound by $n^{1/6}$.

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The function $\triangle(n)$ satisfies

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Conclusion

Best known bounds on $\triangle(n)$:

$$(1 - o(1))n^{5/3} \leq \triangle(n) \leq (1 + o(1))n^{7/4}.$$

$$\triangle(n) = ???$$

Guess: $\triangle(n) = o(n^{7/4})$.

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