* The Eigenvalues of the Graphs $D(4, q)$

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*Joint work with Jason Williford and Shuying Sun

The graphs $D(4,q)$

The graph $D(4,q)$ has $2q^4$ vertices: points and lines in $\mathbb{F}_q^4$ denoted given by

$$p = (p_1, p_2, p_3, p_4), \quad \ell = (\ell_1, \ell_2, \ell_3, \ell_4)$$

with p and ℓ adjacent iff

$$p_2 + \ell_2 = p_1\ell_1, \quad p_3 + \ell_3 = p_1\ell_2, \quad p_4 + \ell_4 = p_2\ell_1.$$

There is an infinite sequence of q-fold covering graphs

$$\cdots \to D(5,q) \to D(4,q) \to D(3,q) \to D(2,q)$$

where $D(k,q)$ is bipartite with $2q^k$ vertices

$$p = (p_1, p_2, p_3, \ldots, p_k), \quad \ell = (\ell_1, \ell_2, \ell_3, \ldots, \ell_k)$$

and the covering maps simply delete the right-most coordinates.

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The graphs $D(4, q)$

$D = D(4, q)$ is naturally regarded as the bipartite incidence graph of a point-line incidence structure: D has adjacency matrix

$$B = \begin{bmatrix} 0 & B_1 \\ B_1^T & 0 \end{bmatrix}$$

where B_1 is the $q \times q$ incidence matrix. The point collinearity graph is $\Gamma = \Gamma(4, q)$ with adjacency matrix $A = B_1 B_1^T - qI_{q^4}$. By finding the eigenvalues of Γ, we may directly infer those of D.

In general $D(k, q)$ is not connected; each of its connected components is denoted $CD(k, q)$.

The spectrum of $CD(4, 2^e)$ is known (Li, Lu and Wang, 2009). For q odd, $D(4, q)$ is connected and thus coincides with $CD(4, q)$. We determine its spectrum.

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Our Main Result

Theorem (M., Williford, Sun (2016))

The graph $D(4,q)$ has eigenvalues $\pm q$, each of multiplicity 1 (unless $q \in \{2,4\}$, when each value $\pm q$ has multiplicity 4). All remaining eigenvalues $\pm\varepsilon$ satisfy $|\varepsilon| \leqslant 2\sqrt{q}$.

Apart from the values $0, \pm\sqrt{q}, \pm\sqrt{2q}$, all remaining eigenvalues have the form

$$\varepsilon = \pm \sum_{a \in \mathbb{F}_q} \zeta^{tr_{\mathbb{F}_q/\mathbb{F}_p} f(a)}, \qquad \zeta = e^{2\pi i/p}$$

for $q = p^e$, $p \neq 3$, where $f : \mathbb{F}_q \to \mathbb{F}_q$ is a cubic polynomial; or of the form

$$\varepsilon = \pm \sum_{a \in \mathcal{T}} \xi^{tr_{R/\mathbb{Z}_9} f(a)}, \qquad \xi = e^{2\pi i/9}$$

for $q = 3^e$, where $R = GR(9,e)$ is the Galois ring of order $q^2 = 9^e$ and characteristic 9; again, $f : R \to R$ is a cubic polynomial. In these cases $|\varepsilon| \leqslant 2\sqrt{q}$ is Hasse's bound.

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Γ as a Cayley graph

$\Gamma = \mathrm{Cay}(G, S)$ is a Cayley graph for a group G of order q^4 with $|G'| = q$, $G'' = 1$:

$$1 \notin S \subset G, \quad s \in S \text{ iff } s^{-1} \in S;$$
$$\text{vertices } g \sim g' \text{ iff } g'g^{-1} \in S.$$

Alas, G is abelian; moreover, the subset $S \subset G$ is *not* 'normal' (a union of conjugacy classes) so the irreducible characters of G do not suffice to express the spectrum of our graphs.

Let $\pi_i : G \to GL_{n_i}(\mathbb{C})$ $(i = 1, 2, \ldots, k)$ be the irreducible ordinary representations of G. Compute the $n_i \times n_i$ matrices

$$\pi_i(S) = \sum_{s \in S} \pi(s).$$

Then the characteristic polynomial of $\Gamma = \mathrm{Cay}(G, S)$ is

$$\phi(x) = \det[xI_{q^4} - A] = \prod_{i=1}^{k} \det[xI_{n_i} - \pi_i(S)]^{n_i}.$$

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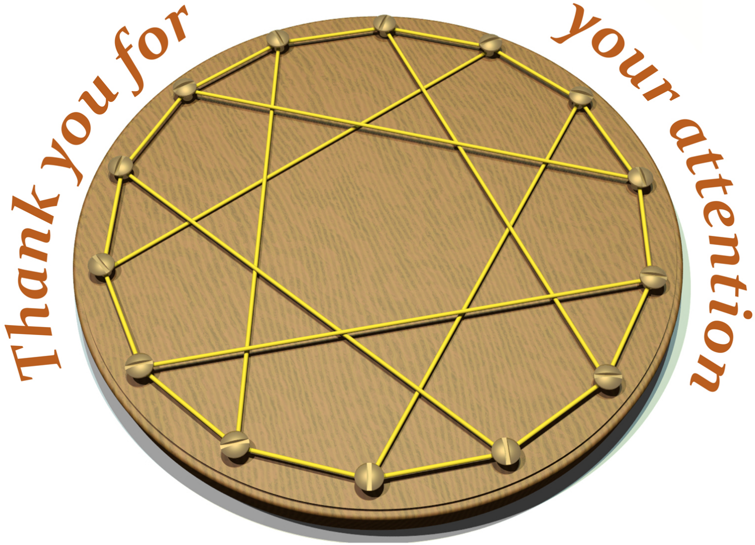
Thank you for
your attention