On the girth of some algebraically defined graphs

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in honor of Willem Haemers, Felix Lazebnik, and Andrew Woldar

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This talk contains joint work with:

- Felix Lazebnik and Jason Williford
- Allison Ganger, Shannon Golden, and Carter Lyons (Supported by NSF #1560222, REU Site: Undergraduate Research in Mathematics, Applied Mathematics, and Statistics at Lafayette College.)

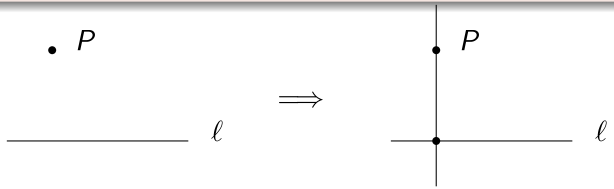

NO SMOKING
Department of Mathematical Sciences

Motivation: What is a Generalized Quadrangle?

Definition

A **generalized quadrangle of order** q is an incidence structure of $q^3 + q^2 + q + 1$ points and $q^3 + q^2 + q + 1$ lines such that...

1. Every point lies on $q + 1$ lines; two distinct points determine at most one line.
2. Every line contains $q + 1$ points; two distinct lines have at most one point in common.
3. If P is a point and ℓ is a line such that P is not on ℓ, then there exists a unique line that contains P and intersects ℓ.



An example: $GQ(1)$ and its point-line incidence graph

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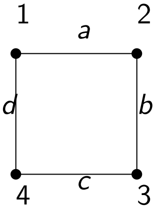

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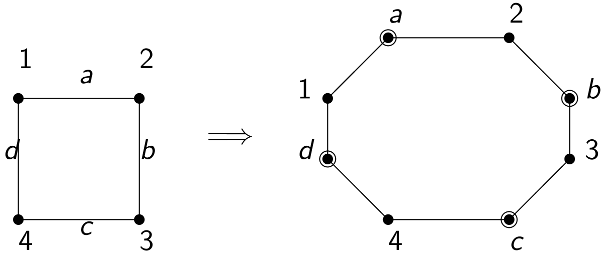
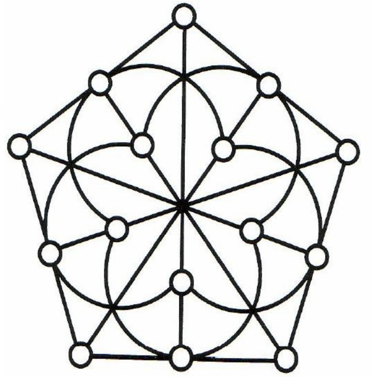


The point-line incidence graph of $GQ(1)$ is 2-regular, has girth eight, and has diameter four.

Example: $GQ(2)$ and its point-line incidence graph



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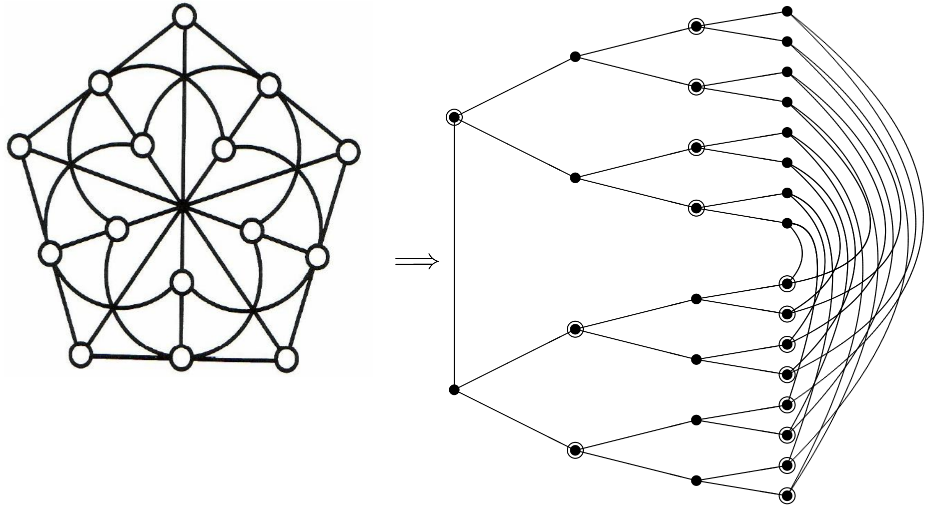
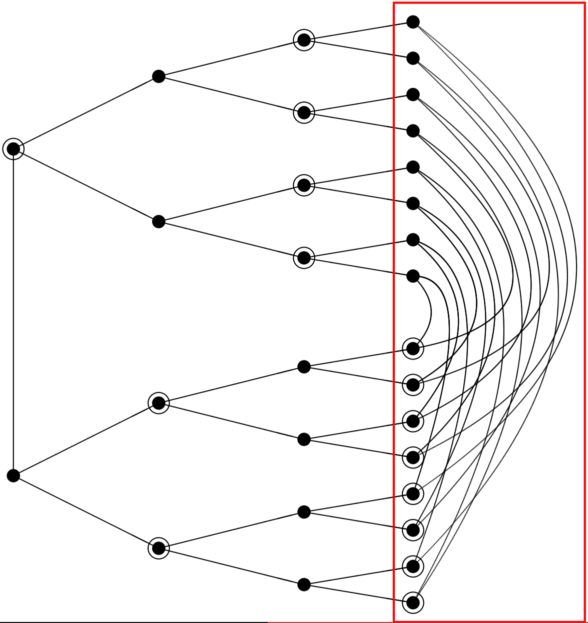
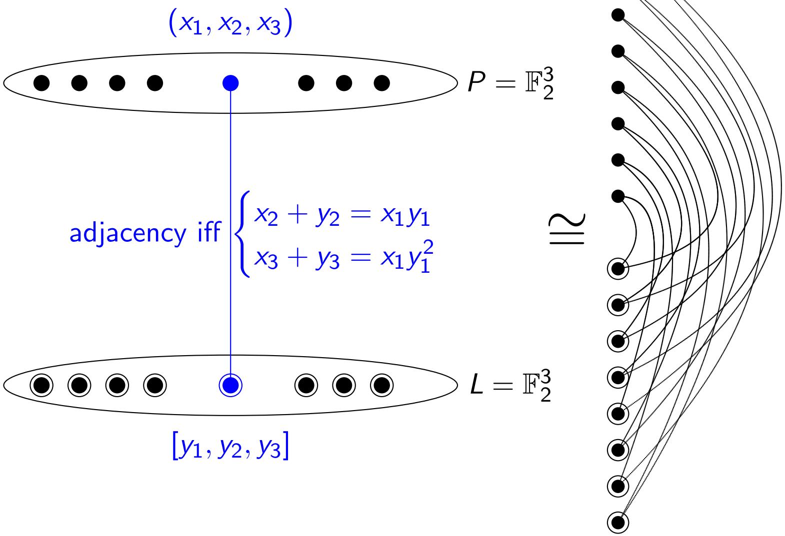


Example: The incidence graph of $GQ(2)$



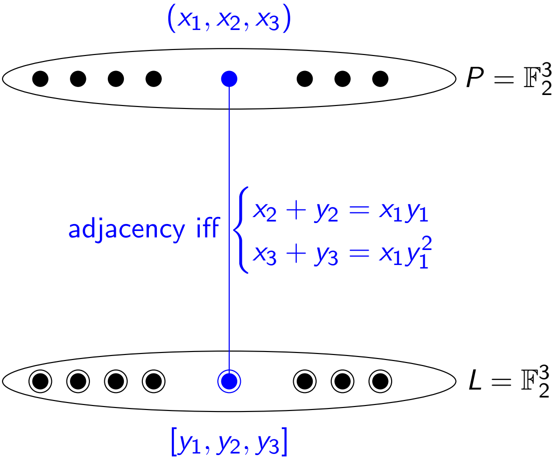
The "Affine Part" of $GQ(2)$ is an ADG

$\Gamma_2(xy, xy^2)$



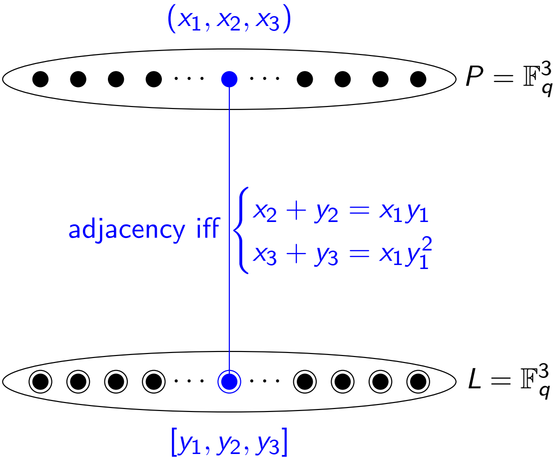
Algebraically Defined Graphs

$\Gamma_2(xy, xy^2)$



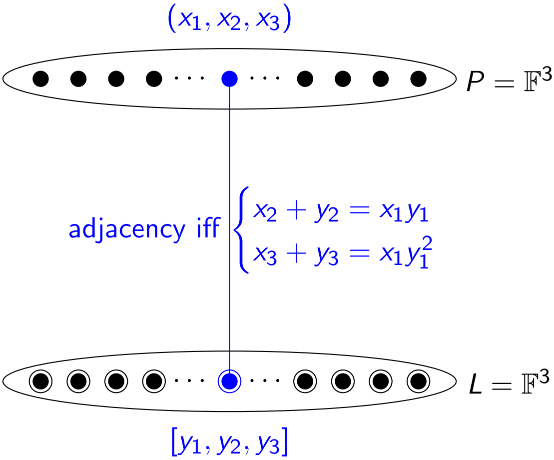
Algebraically Defined Graphs

$\Gamma_q(xy, xy^2)$



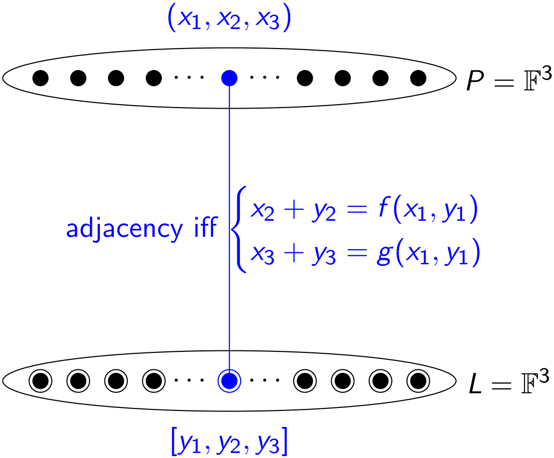
Algebraically Defined Graphs

$\Gamma_{\mathbb{F}}(xy, xy^2)$

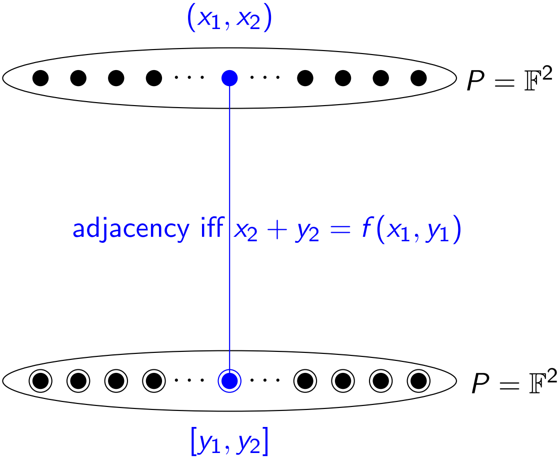


Algebraically Defined Graphs

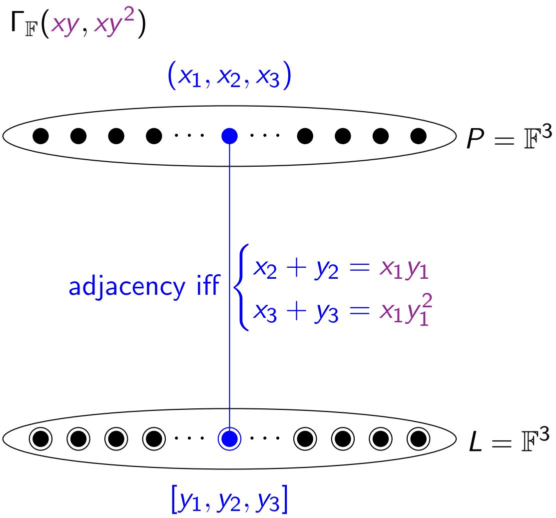
$\Gamma_{\mathbb{F}}(f, g)$



$\Gamma_{\mathbb{F}}(f)$



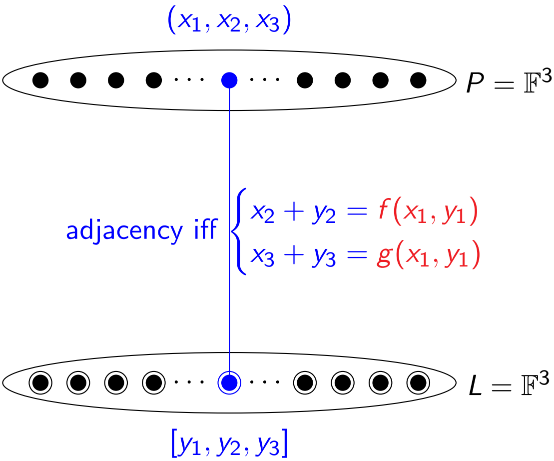
Algebraically Defined Graphs: Motivation



- We know $\Gamma_{\mathbb{F}}(xy, xy^2)$ has girth eight.

Algebraically Defined Graphs: Motivation

$\Gamma_{\mathbb{F}}(f, g)$



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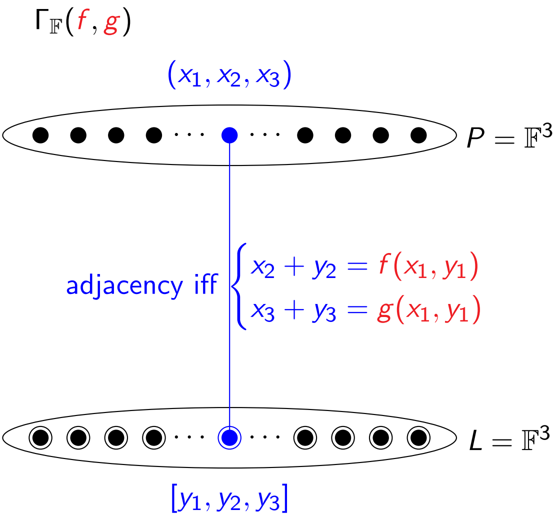

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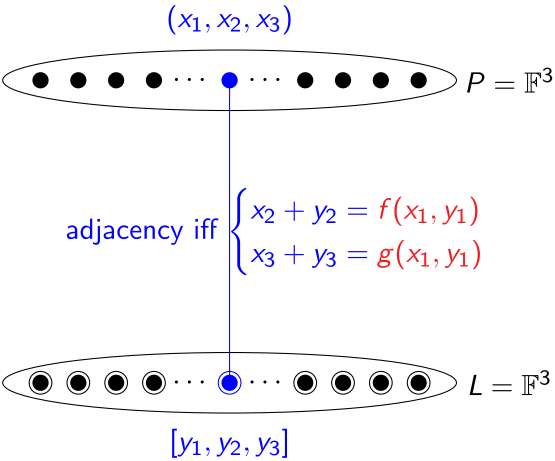

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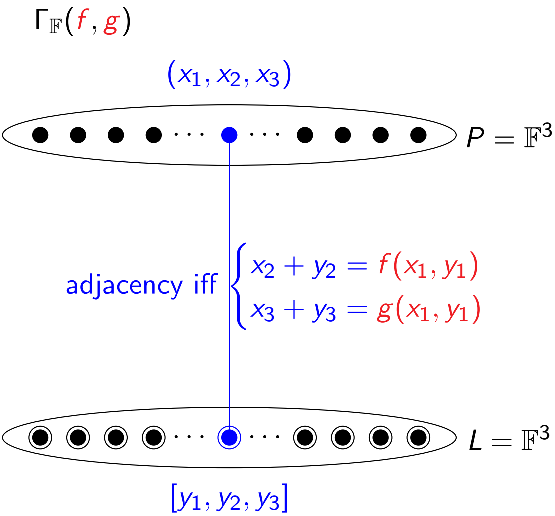

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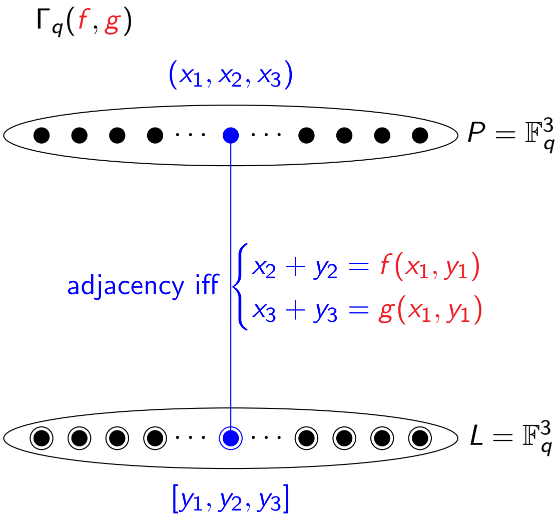
- If $\Gamma_{\mathbb{F}}(f, g)$ has girth eight, must it be isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$?
- If yes, we have an interesting characterization.
- If not, then we might be able to construct a new generalized quadrangle (projective plane with a girth six $\Gamma_{\mathbb{F}}(f) \ncong \Gamma_{\mathbb{F}}(xy)$).

What happens when $\mathbb{F} = \mathbb{F}_q$?



- Where should we start our search over $\mathbb{F}_q$?

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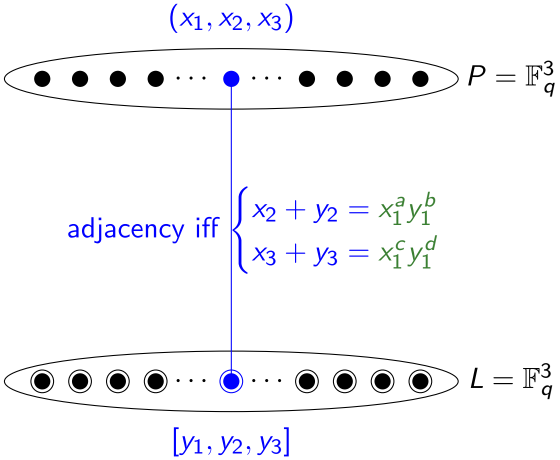
$\Gamma_q(x^a y^b, x^c y^d)$



- Where should we start our search over $\mathbb{F}_q$?
- $\Gamma_q(xy, xy^2)$ has girth eight, so let's begin by studying **monomial graphs**.

What about monomial graphs (in two dimensions)?

Theorem (V. Dmytrenko, F. Lazebnik, R. Viglione; 2005)

Let k, m, k', m' be positive integers and let q, q' be prime powers. Then the graphs $\Gamma_q(x^k y^m)$ and $\Gamma_{q'}(x^{k'} y^{m'})$ are isomorphic if and only if $q = q'$ and the multisets

$$\{\gcd(k, q-1), \gcd(m, q-1)\} \textit{ and } \{\gcd(k', q-1), \gcd(m', q-1)\}$$

are equal.

What about monomial graphs?

Do any monomials f and g produce a girth eight graph that is not isomorphic to $\Gamma_q(xy, xy^2)$?

Conjecture (V. Dmytrenko, F. Lazebnik, J. Williford; 2007)

For any given odd prime power q, $\Gamma_q(xy, xy^2)$ is the unique girth eight algebraically defined monomial graph (up to isomorphism).

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Theorem (V. Dmytrenko, F. Lazebnik, J. Williford; 2007)

Let $q = p^e$ be an odd prime power. Then every monomial graph $\Gamma_q(x^a y^b, x^c y^d)$ of girth at least eight is isomorphic to the graph $\Gamma_q(xy, x^k y^{2k})$, where k is not divisible by p. If $q \geq 5$, then:

1. $((x+1)^{2k} - 1)x^{q-1-k} - 2x^{q-1} \in \mathbb{F}_q[x]$ *is a permutation polynomial of* $\mathbb{F}_q$.
2. $((x+1)^k - x^k)x^k \in \mathbb{F}_q[x]$ *is a permutation polynomial of* $\mathbb{F}_q$.

Results on monomial graphs

Theorem (V. Dmytrenko, F. Lazebnik, J. Williford; 2007)

1. *Let* $q = p^e$ *with* $p \geq 5$ *prime and* $e = 2^m 3^n$ *for integers* $m, n \geq 0$. *Then every girth eight monomial graph* $\Gamma_q(x^a y^b, x^c y^d)$ *is isomorphic to* $\Gamma_q(xy, xy^2)$.
2. *For all odd* q, $3 \leq q \leq 10^{10}$, *every girth eight monomial graph* $\Gamma_q(x^a y^b, x^c y^d)$ *is isomorphic to* $\Gamma_q(xy, xy^2)$.

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Theorem (BGK; 2012)

Let $q = p^e$ *be an odd prime power, with* $p \geq p_0$, *a lower bound that depends only on the largest prime divisor of* e.
Then every girth eight monomial graph $\Gamma_q(x^a y^b, x^c y^d)$ *is isomorphic to* $\Gamma_q(xy, xy^2)$.

More results on monomial graphs

Theorem (X. Hou, S.D. Lappano, F. Lazebnik; 2017)

Let q be an odd prime power. Then every girth eight monomial graph $\Gamma_q(x^a y^b, x^c y^d)$ is isomorphic to $\Gamma_q(xy, xy^2)$.

This means that we'll have to expand our search to algebraically defined graphs where f and g are not both monomials.

Binomial graphs

Theorem (V. Dmytrenko; 2004)

Let $q = p^e$ be an odd prime power, and let $G = \Gamma_q(xy, f)$ be a binomial graph, where $f(x, y) = \beta x^{k_1} y^{m_1} + \alpha x^{k_2} y^{m_2}$, $\alpha\beta \neq 0$.
Then there is a constant C such that for $q > C$, the graph G either has girth six or $G \cong \Gamma_q(xy, x^m y^{2m})$, where $\gcd(m, q-1) = 1$.

Results for more complicated f seem difficult; where else can we look?

Polynomial graphs ... over fields of characteristic zero

In two dimensions:

Theorem (F. Lazebnik and BGK; 2013)

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. Suppose $f \in \mathbb{F}[x, y]$ and the graph $\Gamma_{\mathbb{F}}(f)$ has girth at least six. Then $\Gamma_{\mathbb{F}}(f)$ is isomorphic to $\Gamma_{\mathbb{F}}(xy)$.

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In three dimensions:

Theorem (F. Lazebnik, J. Williford, and BGK; 2017+
$k = m = 1$ case: F. Lazebnik and BGK; 2016)

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, and let k and m be positive integers. Suppose $f \in \mathbb{F}[x, y]$ and the graph $\Gamma_{\mathbb{F}}(x^k y^m, f)$ has girth at least eight. Then $\Gamma_{\mathbb{F}}(x^k y^m, f)$ is isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$.

What does this tell us about the finite fields case?

Theorem

Let q be a power of a prime p, $p \geq 5$. Suppose that $f \in \mathbb{F}_q[x, y]$ has degree at most $p - 2$ with respect to each of x and y. Then there exists a positive integer $M = M(k, m, q)$ such that for all positive integers r:

- *(F. Lazebnik and BGK; 2016)* *every graph $\Gamma_{q^{Mr}}(xy, f)$ of girth at least eight is isomorphic to $\Gamma_{q^{Mr}}(xy, xy^2)$, where $M = M(p)$ is the least common multiple of the integers $1, 2, \ldots p - 2$.*
- *(F. Lazebnik, J. Williford, and BGK; 2017+)* *every graph $\Gamma_{q^{Mr}}(x^k y^m, f)$ of girth at least eight is isomorphic to $\Gamma_{q^{Mr}}(xy, xy^2)$, where k and m are relatively prime to p and $M = M(k, m, q)$ is the least common multiple of the integers $\phi(k)$, $\phi(m)$, 2, 3, …, and $4p - 15$, where ϕ is Euler's totient function.*

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Algebraically defined graphs over $\mathbb{R}$ (in two dimensions)

Theorem (A.J. Ganger, S.N. Golden, C.A. Lyons, BGK; 2017+)

Let $f \in \mathbb{R}[x, y]$. Every graph $\Gamma_{\mathbb{R}}(f)$ has girth at most six.

Theorem (A.J. Ganger, S.N. Golden, C.A. Lyons, BGK; 2017+)

Let $f(x,y) = \sum_{i,j \in \mathbb{N}} \alpha_{i,j} x^i y^j \in \mathbb{R}[x,y]$*. The girth of* $\Gamma_{\mathbb{R}}(f)$ *is as indicated for the following families of* f*:*

Girth 4

- $\sum_{i,j \in \mathbb{N}} \alpha_{i,j} = 0$
- $\sum_{i,j \in 2\mathbb{N}+1} \alpha_{i,j} = 0$
- $\sum_{i,j \in \mathbb{N}} \left(\alpha_{i,j} x^{2i} y^j + \beta_{i,j} x^i y^{2j} \right)$ *such that all non-zero* $\alpha_{i,j} > 0$ *or all non-zero* $\alpha_{i,j} < 0$
- $\alpha_{3,3} x^3 y^3 + \alpha_{2,2} x^2 y^2 + \alpha_{1,1} xy$ *such that* $(\alpha_{2,2})^2 > 3\alpha_{1,1}\alpha_{3,3}$
- *Largest or smallest exponent is even*
- *Coefficients on largest and smallest power terms have opposite signs*
- *Let* p *be the smallest even power of* x*. All terms* $x^i y^j$ *with* $i \leq p$ *are mixed.*

Girth 6

- $\sum_{i,j \in 2\mathbb{N}+1} \alpha_{i,j} x^i y^j$ *such that all non-zero* $\alpha_{i,j} > 0$ *or all* $\alpha_{i,j} < 0$
- $\alpha_{3,3} x^3 y^3 + \alpha_{2,2} x^2 y^2 + \alpha_{1,1} xy$ *such that* $(\alpha_{2,2})^2 \leq 3\alpha_{1,1}\alpha_{3,3}$

Open Questions

- Let $f, g \in \mathbb{F}_q[x, y]$ such that f and g are not both monomials. Classify $\Gamma_q(f, g)$ according to girth.
- Let $f, g \in \mathbb{C}[x, y]$ such that neither f nor g is a monomial. Classify $\Gamma_{\mathbb{C}}(f, g)$ according to girth.
- Let $f, g \in \mathbb{R}[x, y]$. Classify $\Gamma_{\mathbb{R}}(f, g)$ according to girth.