On Erdős–Ko–Rado graphs and Chvátal's conjecture

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Notation and definitions

- $[n] = \{1, \ldots, n\}$, for some positive integer n.
- $2^{[n]} = \{A : A \subseteq [n]\}$
- $\binom{[n]}{r} = \{A \in 2^{[n]} : |A| = r\}$.
- $\mathcal{F} \subseteq 2^{[n]}$ called an **intersecting family on** $[n]$ if for any $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.
- e.g. – $\mathcal{F} = \{\{1,2,3\},\{2,3,4\},\{1,3,4\}\}$ is an intersecting **3-uniform** family on [4].

Intersecting set systems – examples

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  – Size of *largest star* provides a tight upper bound of $\mathbf{2^{n-1}}$ for maximum intersecting subfamilies of $2^{[n]}$.

  – A second *extremal* example: $\mathcal{F} = \{A \subseteq [n] : |A| > \lfloor n/2 \rfloor\}$ (for odd n).

Uniform intersecting families

Theorem (Erdős-Ko-Rado, 61)

If $r \leq n/2$ *and* $\mathcal{A} \subseteq \binom{[n]}{r}$ *is intersecting, then* $|\mathcal{A}| \leq \binom{n-1}{r-1}$*. If* $r < n/2$*, equality holds if and only if* $\mathcal{A} = \{A \in \binom{[n]}{r} : x \in A\}$ *for some* $x \in X$*.*

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Proofs:

- Induction using *shifting* (Erdős et al., '61)
- Cyclic permutations (Katona, '72)
- Kruskal–Katona theorem (Daykin, '74)
- Algebraic approaches
  - Eigenvalues / Hoffman's ratio bound (Godsil, '01)
  - Linear algebra (Füredi et al., '06)

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  - Linear algebra (Füredi et al., '06)
- Shadows of intersecting families (Frankl–Füredi, 2012)
- Injective proof using shifting (Hurlbert–K., 2017)

King Arthur and the Knights of the Round Table

Definition (The Erdős–Ko–Rado problem for cycles)

Let $\mathcal{J}^r(C_n)$ be the family of all r-sized independent sets of the cycle on n vertices. If $\mathcal{F} \subseteq \mathcal{J}^r(C_n)$ is intersecting, how big can it be?

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Theorem (Talbot, 2000)

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$\binom{n-r-1}{r-1}$ **is the size of the *star* centered at any vertex $x \in C_n$.**

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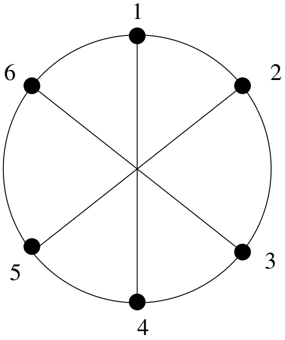
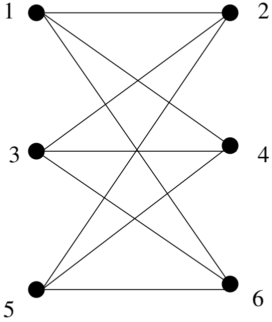
- Can we prove a result analogous to Talbot's theorem for all graphs?
- The answer (obviously) is **No!**

Möbius ladder on $n = 4k + 2$ vertices not r-EKR if $r = \frac{n}{2} - 1$





Mobius ladder on 6 vertices not 2−EKR

Maximum star = {13, 15}. Maximum non−star = {13, 15, 35}

A Conjecture on EKR graphs

Definition ("Minimax" independence number)

$\mu(G)$: minimum size of *maximal* independent set in G.

Conjecture (Holroyd–Talbot, 2005)

If $r \leq \mu(G)/2$, G is r-EKR.

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True for:

- Disjoint union of complete graphs, paths (Holroyd–Spencer–Talbot, '05)
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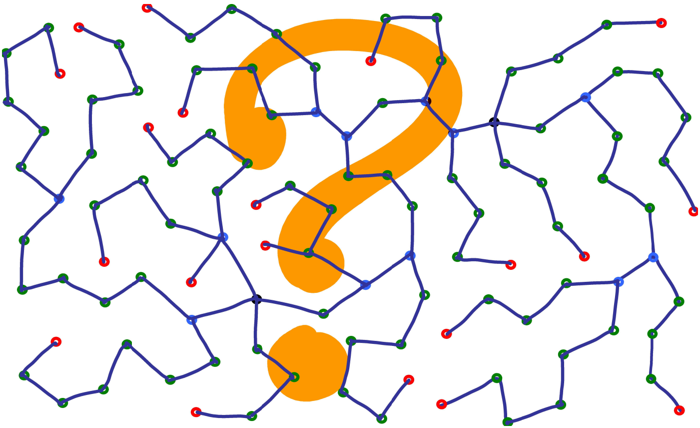
- **Graphs without singletons:**
  - Disjoint union of two cycles (Hilton–Spencer, '09)
  - Chains of complete graphs (Hurlbert–K., '11)
  - Graphs with separation conditions (Borg, '13)

Graphs without singletons



Maximum Stars in Trees

- **Intermediate Question:** Where do the centers of the maximum stars in trees lie?

Theorem (Hurlbert – K., 2011)

For tree T and $1 \leq r \leq 4$, a maximum star of r-independent vertex sets in T is centered at a leaf.

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- Not true when $r \geq 5$. (Baber: 2011, Feghali – Johnson – Thomas, Borg: 2016)

A Counterexample

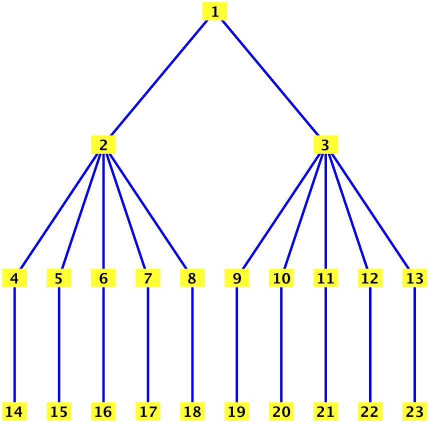

Figure: For $r = 5$, the root vertex beats all leaves

Special classes – Elongated Claws

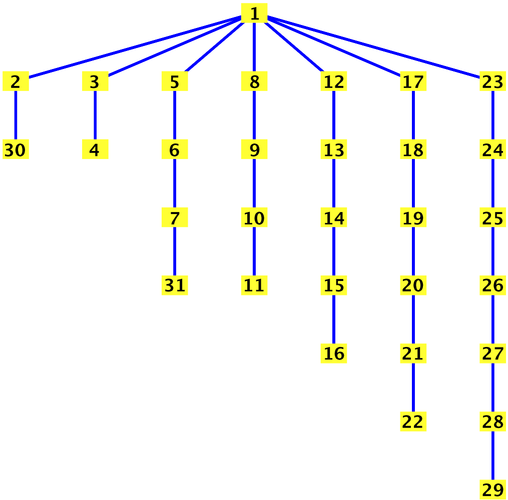

Figure: The 7-claw $C[2, 2, 4, 4, 6, 7, 8]$

Results for Elongated n-Claws

Theorem (Feghali – Johnson – Thomas, 2016)

For $G = C[l_1, \dots, l_n]$:

1. *If $l_1 = 1$ and $r \leq n/2$, G is r-EKR.*
2. *If $l_1 = \cdots = l_n = 2$ and $r \leq \mu(G)/2$, G is r-EKR.*

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Theorem (Hurlbert – K., 2017)

For an elongated n-claw $G = C[l_1, \ldots, l_n]$ with set of leaves $[n]$, where leaf i is at distance l_i from the root, and $1 \leq r \leq \alpha(G)$, there is a maximum star centered at a leaf. Furthermore:

1. *If $l_i < l_j$ and both l_i and l_j are odd, then $|\mathcal{J}_j^r(G)| \leq |\mathcal{J}_i^r(G)|$.*
2. *If $l_i < l_j$ and both l_i and l_j are even, then $|\mathcal{J}_i^r(G)| \leq |\mathcal{J}_j^r(G)|$.*
3. *If l_i is even and l_j is odd, $|\mathcal{J}_i^r(G)| \leq |\mathcal{J}_j^r(G)|$.*

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The question of whether or not elongated *n*-claws obey the Holroyd–Talbot conjecture remains open.

The Holy Grail – Chvátal's conjecture

Definition (Hereditary family)

A family $\mathcal{F} \subseteq 2^{[n]}$ is called hereditary if $F \in \mathcal{F}$ and $G \subseteq F$ implies $G \in \mathcal{F}$.

Conjecture (Chvátal, 1974)

If $\mathcal{F} \subseteq 2^{[n]}$ is hereditary and $\mathcal{G} \subseteq \mathcal{F}$ is intersecting, then there exists an $x \in [n]$ such that $|\mathcal{G}| \leq |\mathcal{F}_x| = \{F \in \mathcal{F} : x \in F\}$.

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– Many of the EKR graphs results stated earlier imply Chvátal's conjecture for subfamilies of the *independence complex* of the corresponding graph class.

Progress on Chvátal's Conjecture

- $\cap\{\max \mathcal{H}\} \neq \emptyset$ (Schonheim, '75)
- $\mathcal{H}$ **left-shifted** for some $x \in [n]$ (Snevily, '92)
- $|\mathcal{I}|_{\max} = |\mathcal{H}|/2$ (Miklos, '84. Wang, '02)
- Union of uniform subfamilies of $\mathcal{H}$, $\mu(\mathcal{H})$ large (Borg, '07)
- $\mathcal{H} \subseteq \binom{[n]}{\leq 3}$. (Sterboul, '74)

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  - $\mathcal{H} \subseteq \binom{[n]}{\leq 3}$, $|\mathcal{I}|_{\max} \geq 31$. (Czabarka, Hurlbert, K.,'17)

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THANK YOU!

Hurlbert-K. Injection for EKR

- $\mathcal{F} = \{124, 126, 146\} \cup \{234, 236, 245, 246, 247, 256, 267, 346, 456, 467\}$

– **left shift**: $6 \to 3$

- $\{123, 124, 134\} \cup \{234, 235, 236, 237, 245, 246, 247, 345, 346, 347\}$

– **left shift**: $4 \to 1$

- $\{123, 124, 125, 126, 127, 134, 135, 136, 137\} \cup \{234, 235, 236, 237\}$

– ***partially* complement** $\mathcal{F}_0$

- $\{123, 124, 125, 126, 127, 134, 135, 136, 137\} \cup \{156, 146, 145, 147\}$