Designs and extremal hypergraph problems

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Abstract

Let $\mathcal{F}$ be a (finite) class of k-uniform hypergraphs, and let $\mathrm{ex}(n, \mathcal{F})$ denote its Turan number, i.e., the maximum size of the $\mathcal{F}$-free, n-vertex, k-uniform hypergraphs. In other words, we consider maximal k-hypergraphs satisfying a local constraint. E.g., a Steiner system $S(n, k, t)$ is just a maximum k-hypergraph with no two sets intersecting in t or more elements.

In this lecture old and new *Turan type problems* are considered. We emphasize constructions applying algebraic/design theoretic tools with some additional twists. Here is a conjecture from the 1980's.

Let $\mathcal{U} = \{123, 456, 124, 356\}$ and $\mathcal{H}$ be a $\mathcal{U}$-free triple system on n vertices. I.e., $\mathcal{H}$ does not contain four distinct members $A, B, C, D \in \mathcal{H}$ such that $A \cap B = C \cap D = \emptyset$ and $A \cup B = C \cup D$, in other words, $\mathcal{H}$ does not have two disjoint pairs with the same union. We conjecture that $|\mathcal{H}| \leq \binom{n}{2}$. Equality can be obtained by replacing the 5-element blocks of an $S(n, 5, 2)$ by its 3-subsets.

The aim of this lecture

Problems and results in Extremal Combinatorics
which are leading to symmetric designs.

TURAN PROBLEM FOR GRAPHS

1. Def's
2. The four-cycle, C_4 and finite projective planes
3. A few other graphs

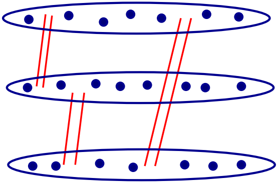
EXTREMAL PROBLEMS ABOUT TRIPLE SYSTEMS

3. Turán's conjectures
4. The Turan number of the Fano plane
5. K_4- and the design $S_2(6, 3, 2)$

***r*-UNIFORM GRAPHS**

6. A problem with an extremum from $S(11, 5, 4)$, $S(12, 6, 5)$
7. A conjecture concerning $S(n, 5, 2)$
8. Grid-free linear hypergraphs
9. Sparse Steiner sytems.

Turán's theorem
Turán type graph problems



K_{p+1} := complete graph,
$T_{n,p}$:= max p-partite graph on n.

Theorem. Mantel (1903) (for K_3)

Turán (1940)

$$e(G_n) > e(T_{n,p}) \Longrightarrow K_{p+1} \subseteq G_n.$$

Unique extremal graph for K_{p+1}.

E.g.: the largest triangle-free graph is the complete bipartite one with $\lfloor n^2/4 \rfloor$ edges.

General question

Given a family $\mathcal{F}$ of forbidden graphs.
What is the maximum of $e(G_n)$ if G_n does not contain subgraphs $F \in \mathcal{F}$?

Notation: $\mathrm{ex}(n, \mathcal{F}) := \max e(G)$

$$\mathrm{ex}(n, K_{p+1}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + O(n).$$

General asymptotics

Erdős-Stone-Simonovits (1946), (1966)

If

$$\min_{F \in \mathcal{F}} \chi(F) = p + 1$$

then

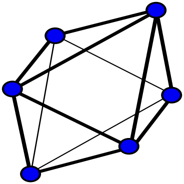
$$\mathrm{ex}(n, \mathcal{F}) = \left(1 - \frac{1}{p}\right) \binom{n}{2} + o(n^2).$$

The asymptotics depends only on the **minimum chromatic number**.

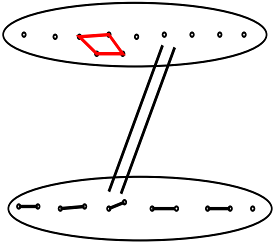
Octahedron Theorem

Erdős-Simonovits

For O_6, $n > n_0$, the extremal graph is a complete bipartite graph $+$ on one side an extremal for C_4 $+$ on the other side a matching.



Excluded: octahedron

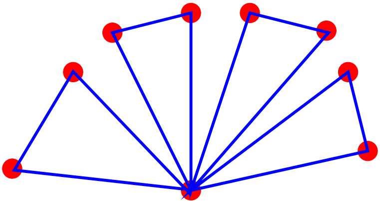


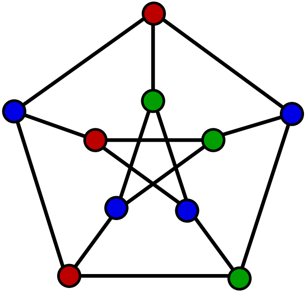
extremal graph

The Problem of Quadrilateral free Graphs

C_4:= four-cycle

$\mathrm{ex}(n, C_4) :=$ max$\{e(G) : G_n$ is quadrilateral-free$\}$.





Fan, F is C_4-free, $\mathrm{ex}(n, C_4) \geq \frac{3}{2}(n-1)$.

Petersen graph is C_4-free, $\mathrm{ex}(10, C_4) \geq 15$.

A simple upper bound

Theorem (Erdős, 1938)

$$\mathrm{ex}(n, C_4) = \Theta(n^{3/2}).$$

Upper bound. G_n is C_4-free $\iff |N(x) \cap N(y)| \leq 1$.
Count the paths of length 2.

$$\binom{n}{2} \geq \text{the number of paths of length 2 in } G = \sum_{x \in V} \binom{\deg(x)}{2}.$$

Use convexity $\binom{n}{2} \geq n\binom{d_{\text{ave}}}{2}$.
This gives

$$n - 1 \geq d_{\text{ave}}(d_{\text{ave}} - 1) \qquad \Rightarrow \qquad \frac{1}{2}(1 + \sqrt{4n-3}) \geq d_{\text{ave}}.$$

A large bipartite C_4-free graph

E. Klein 1938, Reiman 1958.

DEF: bipartite incidency graph of a finite plane, $\mathcal{P}_q$.
Let $n = 2(q^2 + q + 1)$,
The two parts $V(G)$ are P and $\mathcal{L}$
$p \in P$ is adjacent to $L \in \mathcal{L}$ in G if $p \in L$.

$N_G(L_1) \cap N_G(L_2) = L_1 \cap L_2 \quad \Rightarrow \quad |N_1 \cap N_2| \leq 1$
$\Rightarrow \quad G$ is C_4-free.

$e(G) = (q+1)(q^2+q+1) = (1+o(1))\sqrt{\frac{n}{2}}\frac{n}{2}$, hence
$\mathrm{ex}(n, C_4) > (1+o(1))\frac{1}{2\sqrt{2}}n^{3/2}$ for all n.

A polarity of the Desarguesian plane

1	1	1				
1			1	1		
1					1	1
	1		1		1	
	1			1		1
		1	1			1
		1		1	1	

A polarity in the Fano plane.

If $\mathcal{P}$ is Desarguesian, then a π can be defined as $(x, y, z) \leftrightarrow [x, y, z]$. Then two points (x, y, z) and (x', y', z') are joined in G if and only if $xx' + yy' + zz' = 0$.

Many absolute elements

Corollary. If $n = q^2 + q + 1$, $q > 1$, prime(power) then

$$\frac{1}{2}q^2(q+1) \leq \mathrm{ex}(q^2+q+1, C_4) \leq \frac{1}{2}(q^2+q+1)(q+1).$$

A theorem of Baer (1946) states that for every polarity has at least $q+1$ absolute points, $a(\pi) \geq q+1$.
So the lower bound above cannot be improved in this way, the polarity graph cannot have more edges.

Erdős conjectured that the polarity graph is optimal for large q.

Infinitely many exact values

Erdős conj. was proved in the following stronger form.

Theorem 1. (ZF 1983 for $q = 2^\alpha$, ZF 1996 for all q).
Let G be a quadrilateral-free graph on $q^2 + q + 1$ vertices, with $q \neq 1, 7, 9, 11, 13$. Then

$$|\mathcal{E}(G)| \leq \frac{1}{2}q(q+1)^2.$$

Probably holds for all q.

Corollary If $\exists$ a polarity graph with $a(\pi) = q + 1$, then

$$\mathrm{ex}(q^2 + q + 1, C_4) = \frac{1}{2}q(q+1)^2.$$

QUESTION: Extremal graphs?

The extremal graphs

Theorem 2. (ZF)
Let G be a quadrilateral-free graph on $q^2 + q + 1$ vertices, $q \geq 24$, such that $|\mathcal{E}(G)| = \frac{1}{2}q(q+1)^2$.
Then G is the polarity graph.

Symmetric $(q^2+q+2, q+1, 2)$-packings

We use the theory of quasi-designs.
We need results of Ryser (1974), Schellenberg (1974) and Lamken, Mullin and Vanstone (1985), who investigated 0-1 intersecting families on q^2+q+2 points.

DEF: A $(q+1)$-uniform hypergraph $\mathcal{C}$ with q^2+q+2 vertices is called a **special packing** if
– it covers every pair at most once,
– it consists of (q^2+q+2) blocks.

Observation: can yield extremal C_4-free graphs!
Question: Are there infinitely many? (Unsolved).

Conjectures

Erdős conjectured that

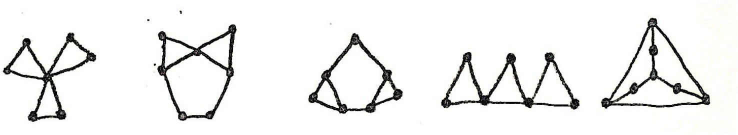
$$|\mathrm{ex}(n, C_4) - \frac{1}{2}n^{3/2}| = O(\sqrt{n}).$$

This conjecture is out of reach at present, even if one knew that the gap between two consecutive primes is only $O(\log^2 p)$.

McCuaig conjectures that
each extremal graph is a subgraph of a polarity graph.
It was proven only for $n \leq 21$.

The case $n \leq 31$

McCuaig (1985) and Clapham, Flockart and Sheehan (1989) determined $\mathrm{ex}(n, C_4)$ and all the extremal graphs for $n \leq 21$. This analysis was extended to $n \leq 31$ by Yuansheng and Rowlinson (1992) by an extensive computer search.



For $n = 7$ there are 5 extremal graphs.
(The last one is the polarity graph, $q = 2$.)

Other values (i.e, $n \neq q^2 + q + 1$)

Firke, Kosik, Nash, and Williford 2013 determined

$$\mathrm{ex}(q^2 + q, C_4) \quad \text{(when } q = 2^\alpha\text{)}.$$

They claimed that they are very close to show that
the extremal graph = polarity graph minus a vertex.

Tait and Timmons 2015
presented a very good construction for $n = q^2 - q - 2$.

No C_4, no C_3.

The points-lines incidency graph of a finite plane gives a bipartite C_4-free graph on
$n = 2(q^2 + q + 1)$ vertices, $(q + 1)(q^2 + q + 1)$ edges.

CONJECTURE. (Erdős and Simonovits)

$$\mathrm{ex}(n, \{C_3, C_4\}) = (1 + o(1))(n/2)^{3/2}.$$

Garnick, Kwong, and Lazebnik 1993 gave the exact value of $\mathrm{ex}(n, \{C_3, C_4\})$ for all n up to 24.
Garnick and Nieuwajaar 1992: for all $n \leq 27$.

Graphs without $K_{2,t+1}$

Thm. (ZF 1996) $t \geq 1$, fixed

$$\mathrm{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{t}n^{3/2} + O(n^{4/3}).$$

Upper bound.
Easy, a special case of Kővári-T. Sós-Turán, 1956.
In G_n any two vertices have $\leq t$ common neighbors.

$$t\binom{n}{2} \geq \text{the number of 2-paths} = \sum_{x\in V}\binom{d(x)}{2} \geq n\binom{2e/n}{2}.$$

Hence

$$e(G) \leq \frac{n}{4}(1 + \sqrt{1+4t(n-1)}).$$

A large graph without $K_{2,t+1}$

Construction.

Let q be a prime power, $(q-1)/t$ is an integer, $\mathbf{F} := \mathbf{F}_q$.
Aim: a $K_{2,t+1}$-free graph G on $(q^2-1)/t$ vertices
with every vertex of degree q or $q-1$.
$H := \{1, h, h^2, \dots, h^{t-1}\}$, $h \in \mathbf{F}$ an element of order t.
The vertices of G are the t-element orbits of

$$(\mathbf{F} \times \mathbf{F}) \setminus (0,0)$$

under the action of multiplication by powers of h.
Two classes $\langle a, b \rangle$ and $\langle x, y \rangle$ are joined by an edge if

$$ax + by \in H.$$

This construction was inspired by examples of Hyltén-Cavallius (1958) and Mörs (1981) given for Zarankiewicz's problem.

Further directions of research

No C_3, no C_4 = girth is at least 5.
Lazebnik, Ustimenko, and Woldar 1995, 1997:

Dense graphs of high girth.

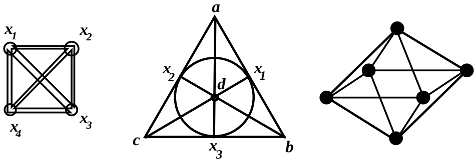
Lazebnik and Woldar 2000, 2001

Graphs defined by systems of equations.

Hypergraph extremal problems

mainly triple systems

3-uniform hypergraphs: **H** $= (V, \mathcal{H})$

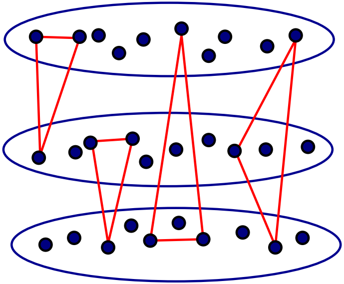


The complete 4-graph, the Fano configuration and the octahedron

Question: $\mathrm{ex}_3(n, \mathbf{H}) = ?$

The famous Turán conjecture (1960)

The following is an extremal structure for $K_4^{(3)}$*:*

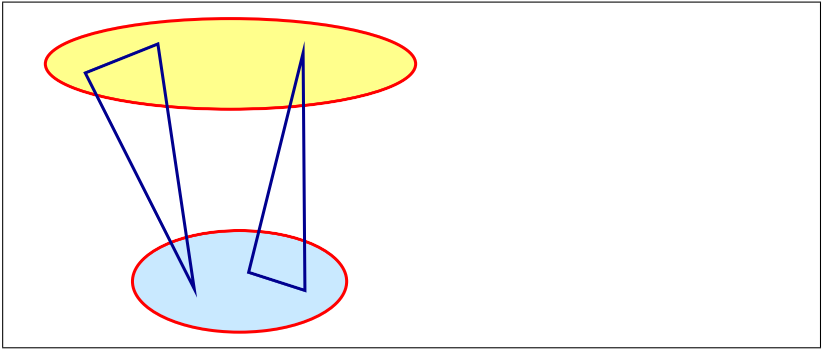


The famous Turán conjecture

If it is true: there is no stability (Brown/Kostochka).

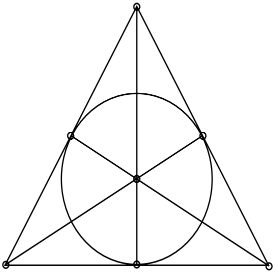
Another conjecture of Turán

The "complete bipartite" 3-graph is extremal for $K_5^{(3)}$.



$$\mathrm{ex}(n, K_5^{(3)}) = (1 + o(1))\frac{3}{4}\binom{n}{3} \quad (?)$$

The Fano configuration, F_7

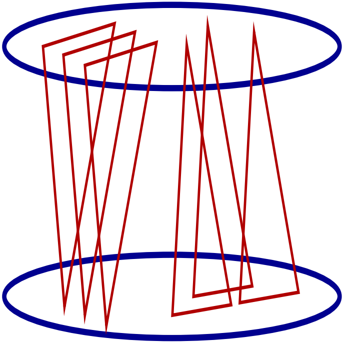


It is a 3-graph of seven edges(=triples) and seven vertices.

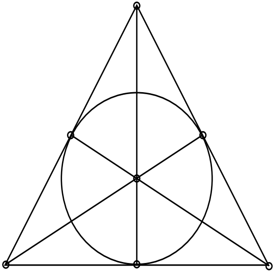
F_7 is 3-chromatic. (The smallest one.)

Conjecture (V. T. Sós (TRUE!))

For $n > n_0$ partition $[n] = X \cup \overline{X}$ with $||X| - |\overline{X}|| \leq 1$ and consider all the triplets containing at least one vertex from both X and $\overline{X}$. Then the 3-uniform hypergraph obtained, $\mathcal{B}(X, \overline{X})$, is extremal for $\mathbf{F}_7$.



Asymptotics for $\mathbf{F}_7$



Theorem [de Caen and ZF 2000].

$$\mathrm{ex}(n, \mathbf{F}_7) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

The Fano-extremal 3-graphs

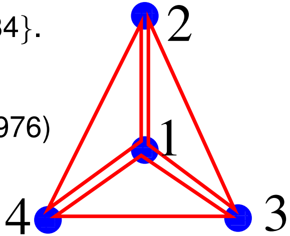
Extremal theorem. [ZF-Simonovits 2002]
If $\mathcal{H}$ is a triple system on $n > n_1$ vertices not containing $\mathbf{F}_7$ *and of maximum cardinality, then $\chi(\mathcal{H}) = 2$. Thus*

$$\mathrm{ex}_3(n, \mathbf{F}_7) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}.$$

Remark. The same was proved independently, in a fairly similar way, by P. Keevash and Benny Sudakov.

Three triples on four vertices

Problem of $\mathrm{ex}_3(n, K_4^-)$.
$K_4^- :=$ 3 triples on 4 points, $\{123, 124, 134\}$.



Question (Brown, Erdős, T. Sós 1973/1976)

$\mathrm{ex}_3(n, K_4^-) = ?$

$\mathcal{F} \subset \binom{[n]}{3}$, $\forall$ 4 elements span at most 2 triples. $\max \mathcal{F} = ?$

Upper bounds:

de Caen $\frac{1}{3}\binom{n}{3} + o(n^3)$, Matthias $\leq \frac{1}{3} - 10^{-26}$, Mubayi $\leq \frac{1}{3} - 3 \times 10^{-5}$, Razborov (2012) et al. $\leq 0.2871\ldots$

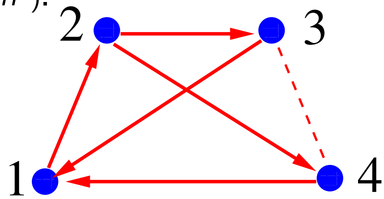
No three triples on four vertices, constructions

Lower bounds.

Erdős, T. Sós 1982 $\geq \frac{1}{4}\binom{n}{3} + o(n^3)$.

Rödl / Frankl & ZF $\geq \frac{1}{4}\binom{n}{3}$.

Take *cyclic triangles*
in a random tournament.

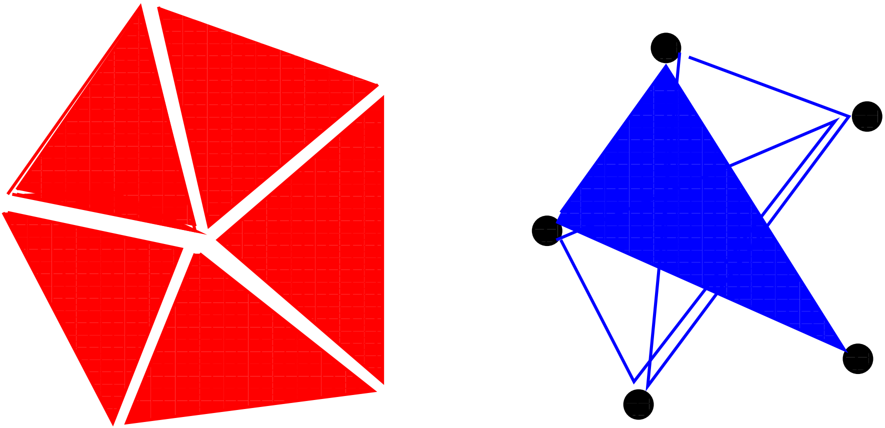


Frankl & ZF: Blow up an $S_2(6,3,2)$.

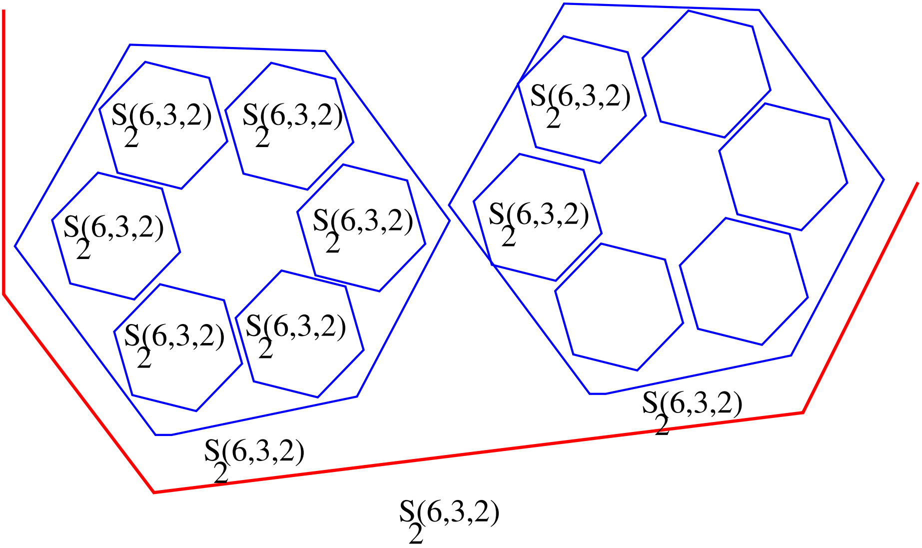
10 triples on 6 vertices yield $10 \times (\frac{n}{6})^3 = \frac{n^3}{21.6}$ triples. Iterate!

Conjecture $\mathrm{ex}_3(n, K_4^-) = \frac{2}{7}\binom{n}{3} + o(n^3)$?

Definition of the $S_3(6,3,2)$ triple system



Blowing up and iterating the 6 groups



A Conjecture of Erdős and Sós

$\mathcal{H} \subset \binom{[n]}{3}$ and every link is bipartite then

$$|\mathcal{H}| \leq (1 + o(1))n^3/24.$$

Link has no triangle $\iff$ there is no $H(4, 3)$.

Link is bipartite $\implies$ there is no $H(4, 3)$.

A construction:

Take a random tournament on $[n]$.

$\mathcal{H} :=$ { the vertex sets of directed triangles }.

A problem with an extremum from the Witt designs

$\Sigma_k := \{A, B, C : A \triangle B \subseteq C, |A \cap B| = k - 1, A \cap B \cap C = \emptyset\}$

Theorem (Bollobas for $k = 3$, Sidorenko 1987 for $k = 4$, Frankl, ZF 1989 for $k = 5, 6$. Asked by de Caen.)

$$\mathrm{ex}(n, \Sigma_3) \le (n/3)^3$$

$$\mathrm{ex}(n, \Sigma_4) \le (n/4)^4$$

$$\mathrm{ex}(n, \Sigma_5) \le \frac{6}{11^4} n^5$$

with equality holding for $n > n_0$, $11|n$,

$$\mathrm{ex}(n, \Sigma_6) \le \frac{11}{12^5} n^5$$

with equality holding for $n > n_0$, $12|n$.

Construction from Witt designs

Take an $S(11,5,4)$ Steiner system $\mathcal{W}_{11}$.
It has $\frac{\binom{11}{4}}{\binom{5}{4}} = 66$ quintuples on 11 elements such that every 4-tuple is covered exactly once.
Especially, $|A \cap B| \leq 3$ holds for every two sets.

Let $11|n$ and $[n] = X_1 \cup X_2 \cup X_{11}$, $|X_i| = n/11$.
Take the blow up of $\mathcal{W}_{11}$ with parts $X_1, \ldots, X_{11}$, i.e.,
$\mathcal{F} := \{A \subset [n] : |A| = 5 \text{ and } \{i : A \cap X_i \neq \emptyset\} \in \mathcal{W}_{11}\}$.

$\mathcal{F}$ has no Σ_5, and $|\mathcal{F}| = 66(\frac{n}{11})^5$.

Similar constructrion for $k = 6$ from the other small Witt design.
$\mathcal{W}_{12}$ is a $S(12,6,5)$ design, it has $\frac{\binom{15}{5}}{\binom{6}{5}} = 132$ six-tuples.
Its blow-up contains $132(\frac{n}{12})^6$ edges contains no Σ_6.

Better results

$$T_k := \{\{1,2,3,\ldots,k\},\{1,2,3,\ldots,k-1,k+1\},\\ \{k,k+1,\ldots,2k-1\}\}.$$

$T_k \in \Sigma_k$.

Observation (Frankl, ZF 1989)

$$\mathrm{ex}(n,\Sigma_k) \le \mathrm{ex}(n, T_k) \le \mathrm{ex}(n,\Sigma_k) + O_k(n^{k-1}).$$

Theorem

$$\mathrm{ex}(n,\Sigma_k) = \mathrm{ex}(n, T_k)$$

for $n > n_0(k)$ and $k=3$ by Frankl, ZF 1983, for $k=4$ by Pikhurko 2008, for $k=5,6$ by Norin and Yepremyan 2017. (Stability, 31 pages.)

Disjoint union free families

$$\text{No } A \cup B = C \cup D \text{ with } A \cap B = \emptyset = C \cap D$$

(for four distinct A, B, C, and D).

Problem (Erdős, 1970's)
$\mathcal{F} \subset \binom{[n]}{k}$ with no two pairs of disjoint members with the same union.

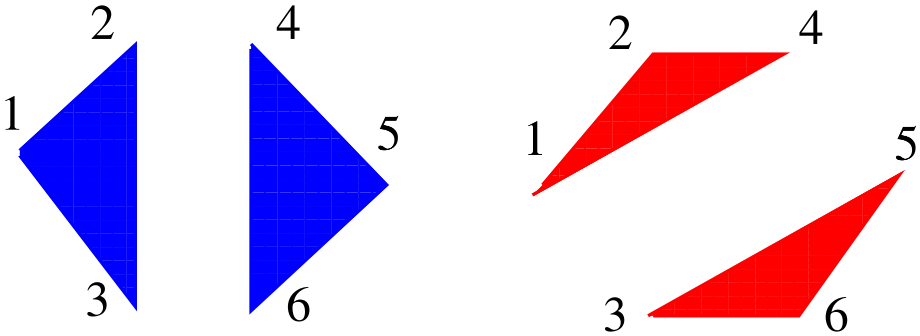
$$\mathbf{DU}_k(n) := \max |\mathcal{F}| =?$$

Theorem (ZF, 1983) $\binom{n-1}{k-1} \leq \mathbf{DU}_k(n) \leq \frac{7}{2}\binom{n}{k-1}$.

$7/2$ was improved to 3 By Mubayi and Verstraëte (2004) and to $13/9$ by Pikhurko and Verstraëte (2009) (for $n > n_0$).

Conjecture $\mathbf{DU}_3(n) = \binom{n}{2}$ for inf' many times.

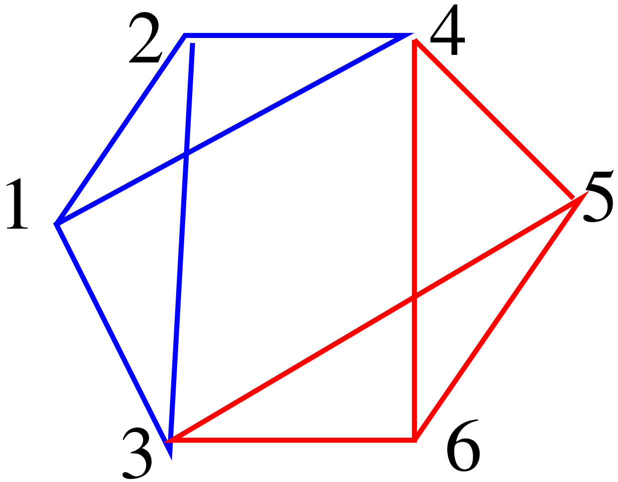
Disjoint triples with the same union



$\mathcal{U} := \{123, 124, 356, 456\}$.

$\mathrm{ex}_3(n, \mathcal{U}) = ?$

Four triples obtained from disjoint pairs

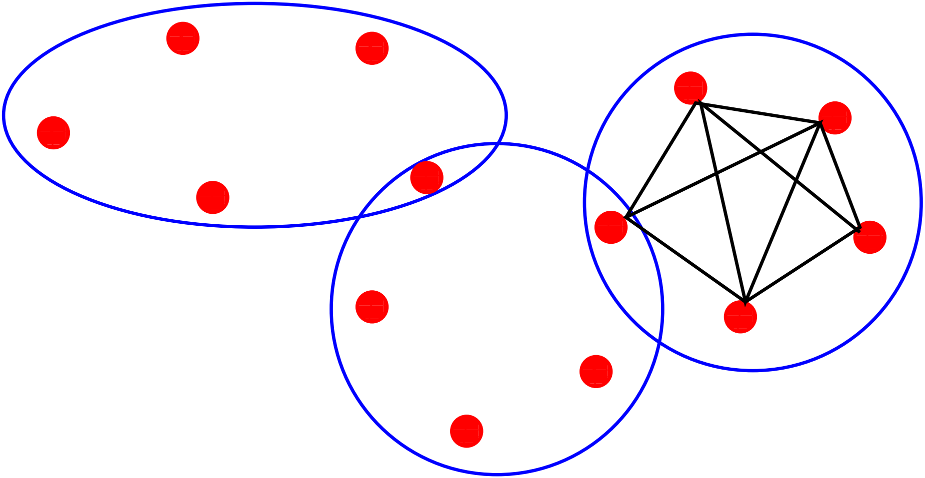


123, 124, 356, 456.

A disjoint-union-free triple system

Start with a $S(n, 5, 2)$ and

fill up the 5-tuples with $(\binom{n}{2}/\binom{5}{2}) \times \binom{5}{3} = \binom{n}{2}$ triples.



No triangles, no $r \times r$ grids

$\mathbf{UF}_r(n,r) := \max |\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{r}$ such that

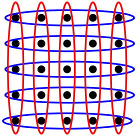
$$A_1 \cup A_2 \cup \cdots \cup A_r = B_1 \cup \cdots \cup B_r$$

and $A_i, B_j \in \mathcal{F}$ imply $\{A_1, A_2, \ldots, A_r\} = \{B_1, \ldots, B_r\}$.

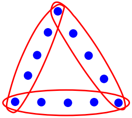
Theorem (ZF & Ruszinkó 2013)

There exists a $\beta = \beta(r) > 0$ *such that for all* $n \geq r \geq 4$

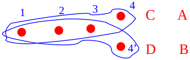
$$n^2 e^{-\beta_r \sqrt{\log n}} < \mathrm{ex}(n, \{\mathbb{I}_{\geq 2}, \mathbb{T}_3, \mathbb{G}_{r\times r}\}) \leq \mathbf{UF}_r(n,r) \leq \frac{n(n-1)}{r(r-1)}.$$



$r \times r$ grid, $\mathbb{G}_{r\times r}$



Triangle $\mathbb{T}_r$



a member of $\mathbb{I}_{\geq 2}$

CONJECTURE: $\mathbf{UF}_r(n,r) = o(n^2)$ for all $r \geq 3$.

What other small substructures can be avoided?

Grid-free linear hypergraphs

Corollary (Grid-free packings)

For $r \geq 4$ there exists a real $c_r > 0$ such that there are linear r-uniform hypergraphs $\mathcal{F}$ on n vertices containing no grids and

$$|\mathcal{F}| > \frac{n(n-1)}{r(r-1)} - c_r n^{8/5}.$$

$$\frac{n(n-1)}{r(r-1)} - c_r n^{8/5} < \mathrm{ex}_r(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{r\times r}\}) \leq \frac{n(n-1)}{r(r-1)}$$

holds for every $n, r \geq 4$.

Conjecture (grid-free Steiner systems)

$\exists$ an $n(r)$ such that, for every admissible $n > n(r)$ (this means that $(n-1)/(r-1)$ and $\binom{n}{2}/\binom{r}{2}$ are both integers) there exists a grid-free $S(n, r, 2)$.

Grid-free triple systems

In the case of $r = 3$ with probabilistic method we only have

$$\Omega(n^{1.8}) \le \mathrm{ex}_3(n, \{\mathbb{I}_{\ge 2}, \mathbb{G}_{3\times 3}\}) \le \frac{1}{6}n(n-1),$$

Conjecture

The asymptotic $\mathrm{ex}_3(n, \{\mathbb{I}_{\ge 2}, \mathbb{G}_{3\times 3}\}) = \Theta(n^2)$ *holds for* $r = 3$*, too.*

There are infinitely many Steiner triple systems avoiding $\mathbb{G}_{3\times 3}$.

There is a large literature of the existence of Steiner triple systems avoiding certain (small) subconfigurations.

A Conjecture on Sparse Steiner systems

A $\mathrm{STS}(n)$ is called *e-sparse* if
every set of e distinct triples span at least $e + 3$ points.

Conjecture (Erdős 1973)

For every $e \geq 2$ there exists an $n_0(e)$ such that if $n > n_0(e)$ and n is admissible (i.e., $n \equiv 1 \text{ or } 3 \pmod 6$), then there exists an e-sparse $\mathrm{STS}(n)$.

Solution for $e = 4$ by Brouwer 1977, Murphy 1990, 1993, Ling and Colbourn 2000, Grannell, Griggs and Whitehead 2000.

$e = 5$ by Colbourn, Mendelsohn, Rosa, and Širáň 1994, Fujiwara 2006 and Wolfe 2005, 2008.

Infinitely many constructions for 6-sparse by Forbes, Grannell and Griggs 2007, 2009.

Teirlinck writes in his 2009 review “currently no nontrivial example of a 7-sparse Steiner triple system is known”.

The end

THANK YOU