Small regular graphs of girth 5

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$(k,5)$-Cages

Definition

A (k,g)-graph is a k-regular graph with girth g.
A (k,g)-cage is a (k,g)-graph with the least possible number of vertices denoted $n(k,g)$.

- We focus on $g=5$.

  - Moore Bound: $n(k,5) \geq 1+k^2$.

  - Moore Cage: $n(k,5) = 1+k^2$.

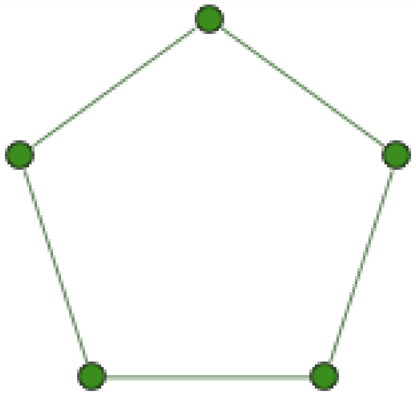
Moore Cages of girth 5 can exist only for:

- $k = 2$, the pentagon;
- $k = 3$, Petersen Graph;
- $k = 7$, Hoffman-Singleton Graph;
- $k = 57$, not constructed until now: Open problem.

$g = 5$

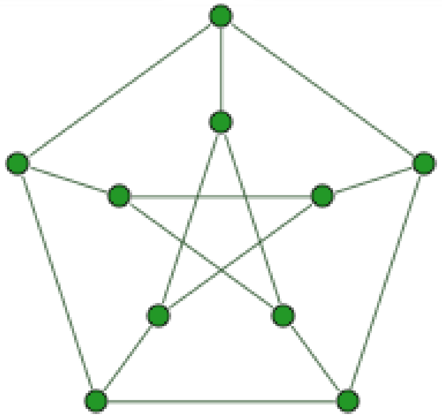
- $The\ order\ of\ an\ (k, 5) - cage\ is\ at\ least$ $n(k, 5) \geq 1 + k^2$

$k = 2$



$5\ vertices$

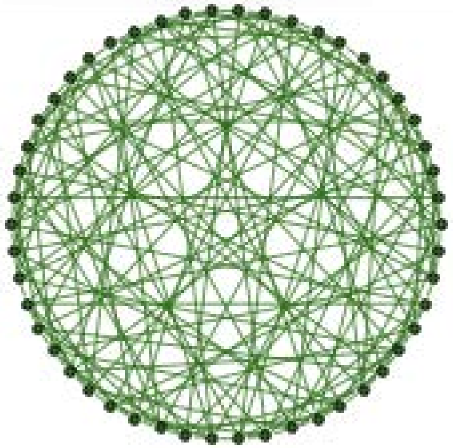
$k = 3$



Petersen

$10\ vertices$

$k = 7$



Hoffman-Singleton

$50\ vertices$

$g = 5$

- $The\ order\ of\ an\ (k, 5) - cage\ is\ at\ least$ $n(k, 5) \geq 1 + k^2$

$\boldsymbol{n(57, 5) = 3250}$

Unknown until now, open question

What about $k \neq 2, 3, 7, 57$

Brown 1967

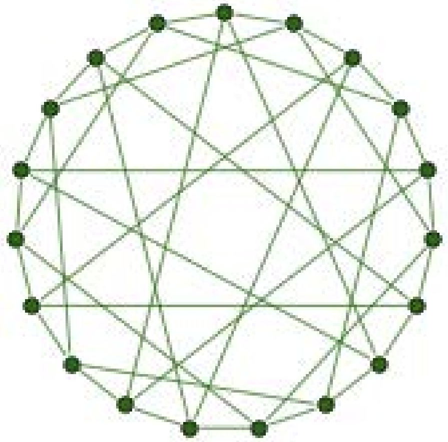
Theorem: *There is no* $(k, 5)$*-graph on* $k^2 + 2$ *vertices. Then:*

$$n(k, 5) \geq k^2 + 3.$$

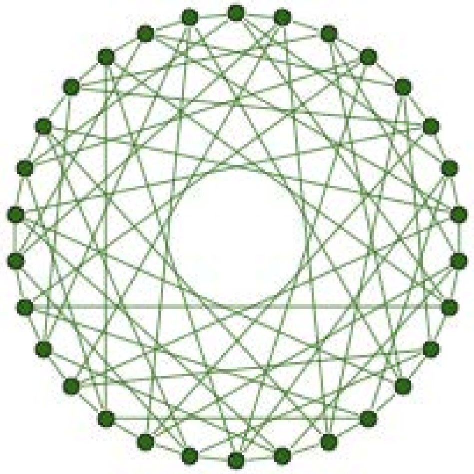
- $k = 4$: Robertson Graph is $(4, 5)$-graph on $19 = 4^2 + 3$ vertices. Then it is a cage, further it is unique.
- $k = 5$: There are 4 non isomorphic $(5, 5)$-graphs on $30 = 5^2 + 5$ vertices which are cages.
- $k = 6$: There is a unique $(6, 5)$-graph on $40 = 6^2 + 4$ vertices; it can be obtained by deleting one Petersen graph from Hofmann-Singleton graph.

$g = 5$

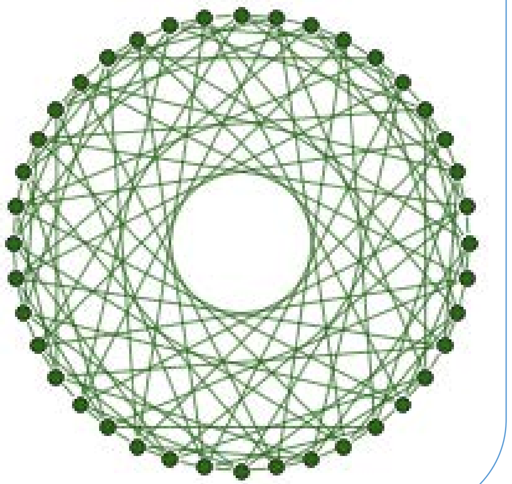
$k = 4$



$k = 5$



$k = 6$



Robertson

$19\ vertices$

Foster

$30\ vertices$

O'Keefe-Wong

$40\ vertices$

Known cages of girth 5

k	$n(k,5)$	
2	5	C_5
3	10	Petersen graph
4	19	Robertson graph
5	30	4 non isomorphic graphs
6	40	
7	50	Hoffmann-Singleton

Better upper bound on $n(k,5)$ for $8 \leq k \leq 12$.

Better upper bound for $n(k,5)$ is denoted by $rec(k,5)$, i.e., $n(k,5) \leq rec(k,5)$.

Current values of $rec(k,5)$ for $8 \leq k \leq 12$.

k	Lower bound	$rec(k,5)$	Due to
8	67	80	Royle, Jørgensen 2005
9	84	96	Jørgensen 2005
10	103	124	Exoo
11	124	154	Exoo
12	147	203	Exoo

- Untouched!

Objective:

To improve $rec(k, 5)$ **at least for** $k \geq 13$**.**

Our contribution

[AABL12] Abreu, Araujo-Pardo, B., Labbate, Families of small regular graphs of girth 5, Discrete Math. 312 (2012) 2832–2842.

[AABB17] Abajo, Araujo-Pardo, B., Bendala, New small regular graphs of girth 5, Discrete Math. 340(8) (2017) 1878–1888.

[ABBs] Abajo, B., Bendala, Regular graphs of girth 5 from elliptic semiplanes, Submitted.

Results

Theorem

For every odd prime power $q \geq 13$ and $k \leq q+3$

- $n(k,5) \leq 2(qk-3q-1)$ [AABL12].
- If $53 \leq k \leq q+6$, then $n(k,5) \leq 2(qk-5q-k+5)$ [AABB17].
- If $68 \leq k \leq 2^{\alpha}+6$, then $n(k,5) \leq 2^{\alpha+1}(k-6)$ [AABB17].

The above results are an improvement of:

- $n(k,5) \leq 2(qk-2q-k+2)$ [Jørgensen (2005)].

Current and our new values of $rec(k,5)$ for $14 \leq k \leq 31$.

k	$rec(k,5)$	Due to	**New** $rec(k,5)$ **AABB17**
14	284	AABL12	
15	310	AABL12	
16	336	Jørgensen 2005	
17	448	Schwenk 2008	*436*
18	480	Schwenk 2008	*468*
19	512	Schwenk 2008	*500*
20	572	AABL12	*564*
21	682	AABL12	*666*
22	720	Jørgensen 2005	*704*
23	918	AABL12	
24	964	AABL12	
25	1010	AABL12	
26	1056	AABL12	
27	1198	AABL12	
28	1248	FUNK (2009), AABL12	
29	1402	AABL12	
30	1456	AABL12	
31	1622	AABL12	

Current and our new values of $rec(k,5)$ for $32 \leq k \leq 52$.

k	$rec(k,5)$	Due to	**New** $rec(k,5)$ **AABB17**
32	1680	Jørgensen	*1624*
33	1856	Funk	*1680*
34	1920	Jørgensen	*1800*
35	1984	Funk	*1860*
36	2048	Funk	*1920*
37	2514	AABL12	*2048*
38	2588	AABL12	*2448*
39	2662	AABL12	*2520*
40	2736	Jørgensen	*2592*
41	3114	AABL12	*2664*
42	3196	AABL12	*2736*
43	3278	AABL12	*3040*
44	3360	Jørgensen	*3120*
45	3610	AABL12	*3200*
46	3696	Jørgensen	*3280*
47	4134	AABL12	*3360*
48	4228	AABL12	*3696*
49	4322	AABL12	*4140*
50	4416	Jørgensen	*4232*
51	4704	Jørgensen	*4324*
52	4800	Jørgensen	*4416*

A curiosity

Theorem 1

Given an integer $k \geq 53$, let q be the lowest odd prime power such that $k \leq q + 6$. Then $n(k, 5) \leq 2(q - 1)(k - 5)$.

Corollary

k	$rec(k,5)$	Due to	**New** $rec(k,5)$ **AABB17**
57	6374	AABL12	*5408*

Further new results contained in [ABBs]

Starting point

Moore $(k, 6)$-cages of order $2(k^2 - k + 1)$ are known to exist for $k = q + 1$, q a prime power. They are denoted Γ_q, and they are the incidence graph of a projective plane of order q.

Elliptic planes type	q-regular	Construction from Γ_q
C	q^2 points q^2 lines	$\mathcal{C}_q = \Gamma_q - N(p) - N(l)$ for $p \sim l$
L	$q^2 - 1$ points $q^2 - 1$ lines	$\mathcal{L}_q = \Gamma_q - N[p] - N[l]$ for $p \not\sim l$

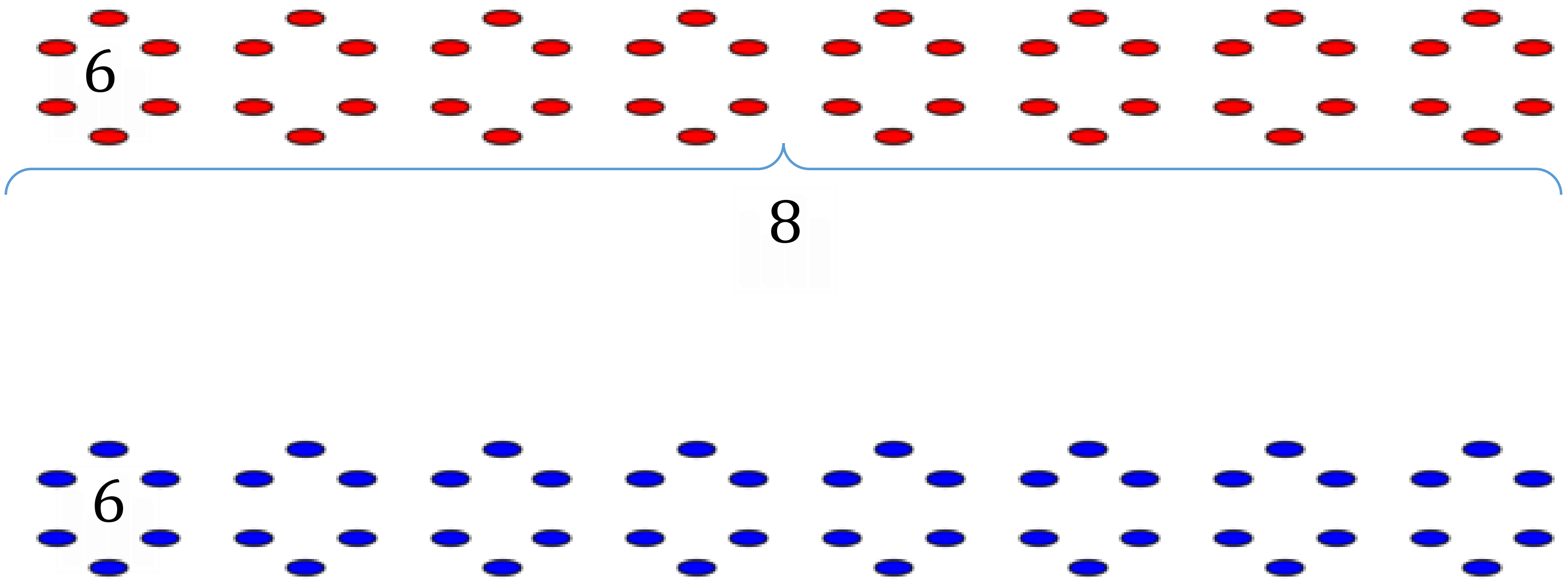
Working with $\mathcal{L}_q$:

- $\mathcal{L}_q = (L, P)$ is bipartite such that
- $L = \cup_{a=0}^{q} L_a$ and $P = \cup_{x=0}^{q} P_x$, $|L_a| = |P_x| = q - 1$,
- L_a and P_x are 4-independent;
- (L_a, P_x) is a matching of $q - 1$ edges for $a \neq x$, and (L_a, P_x) is an empty set of edges.

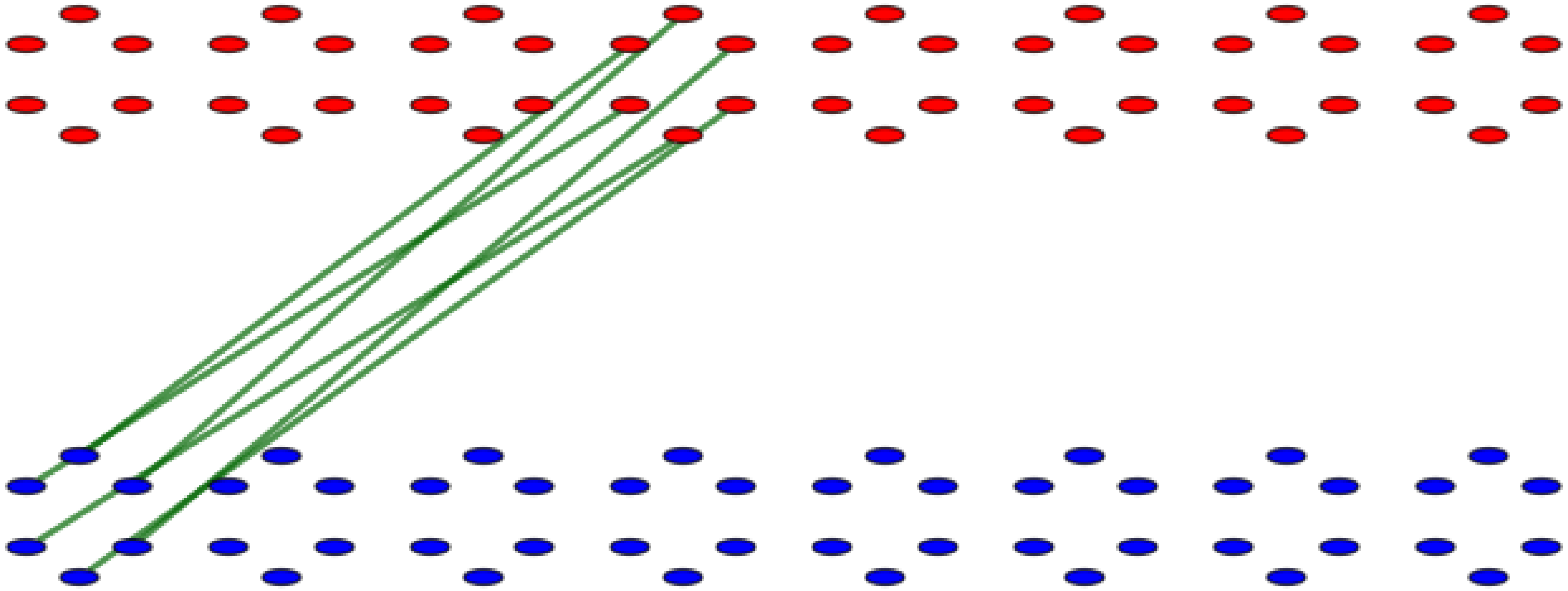
The idea is to add new edges between vertices in the sets L_i and P_j of the graph L_q, increasing its regularity and keeping the girth of the resulting graph in at least 5.

- Label the vertices in L_a as $\ell[a, i]$, and the vertices in P_x as $p(x, i)$, for $i \in \mathbb{F}_q^*$

$$q = 7 \longrightarrow \begin{array}{l} 2\ (q+1)(q-1)\ \textit{vertices} \\ 2 \cdot 8 \cdot 6 = 96\ \textit{vertices} \end{array}$$



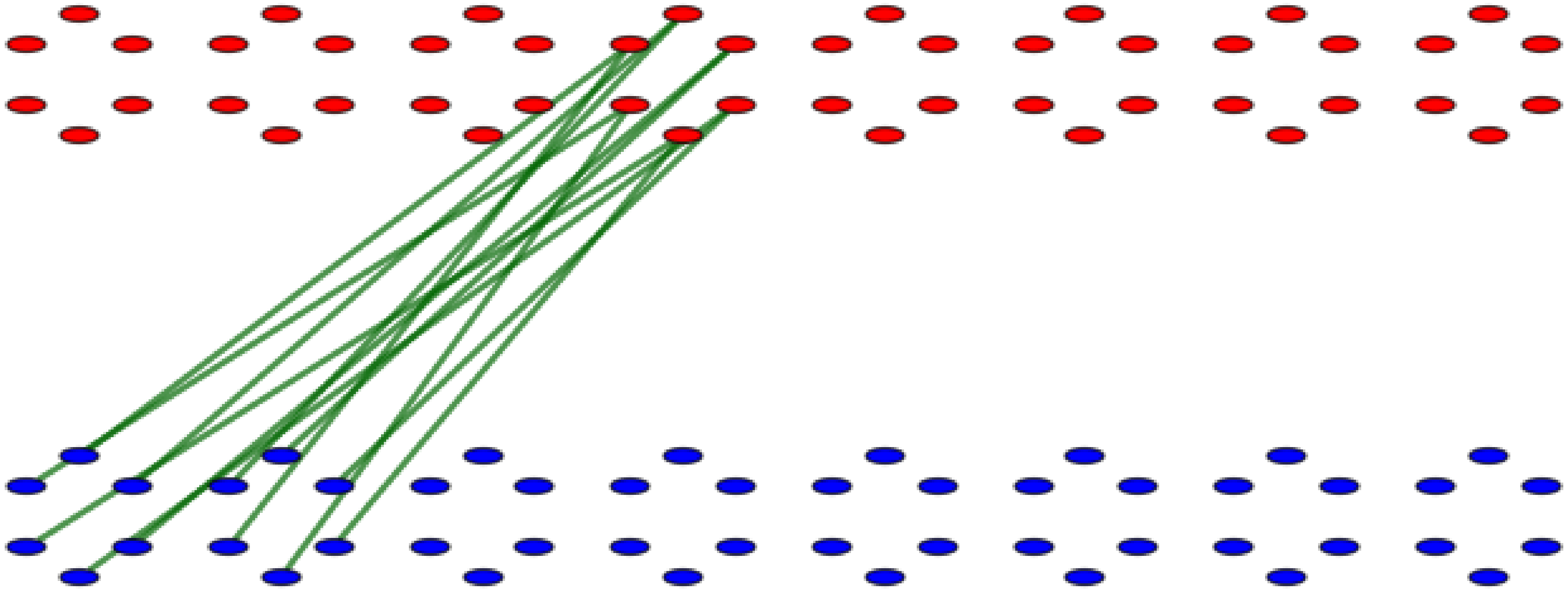
$$q = 7 \longrightarrow \begin{array}{c} 2\ (q+1)(q-1)\ \textit{vertices} \\ 2 \cdot 8 \cdot 6 = 96\ \textit{vertices} \end{array}$$



$$q = 7 \longrightarrow$$

$$2\ (q+1)(q-1)\ \textit{vertices}$$

$$2 \cdot 8 \cdot 6 = 96\ \textit{vertices}$$



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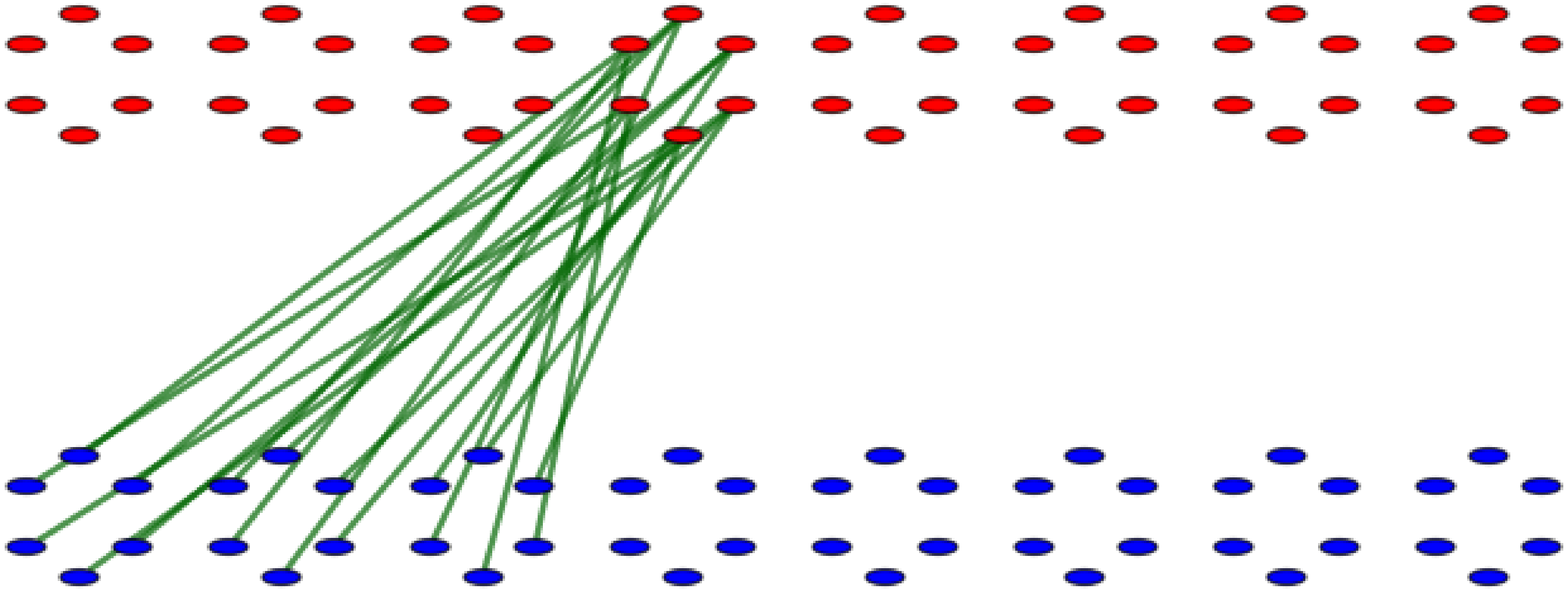

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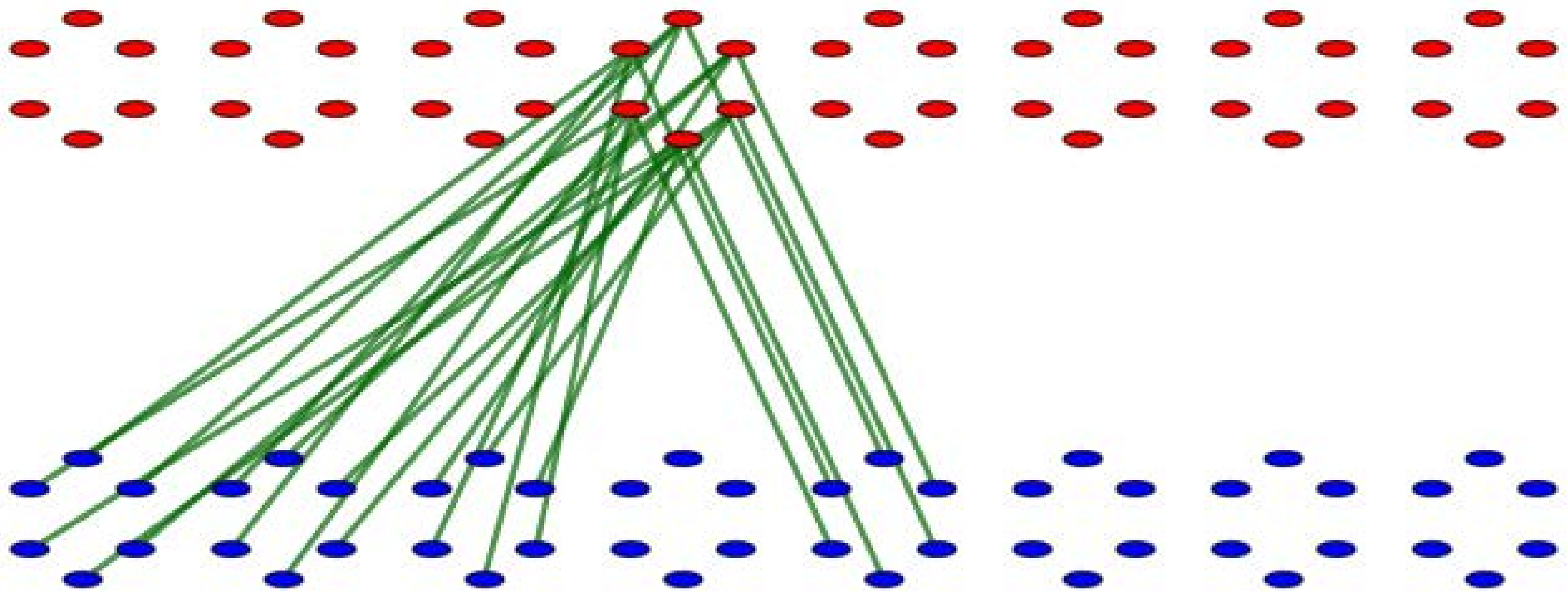
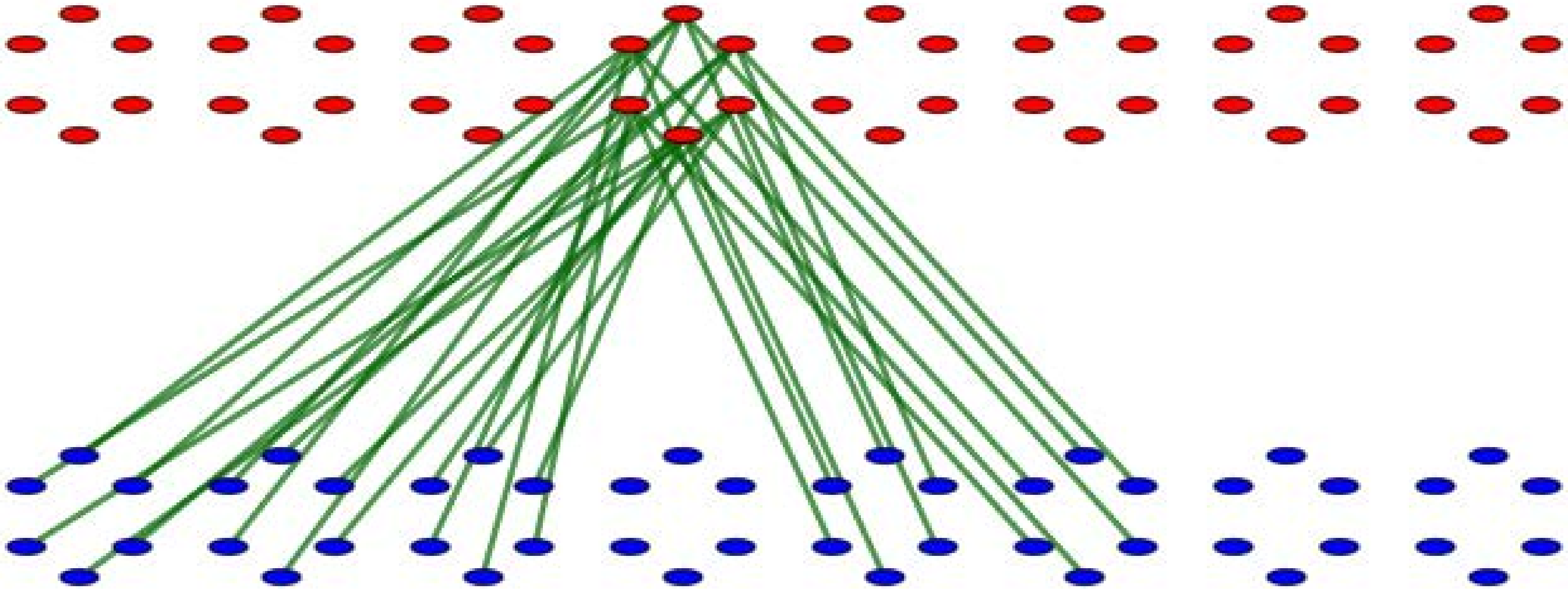


$$q = 7 \longrightarrow \begin{array}{c} 2\ (q+1)(q-1)\ \ vertices \\ 2\cdot 8\cdot 6 = 96\ vertices \end{array}$$



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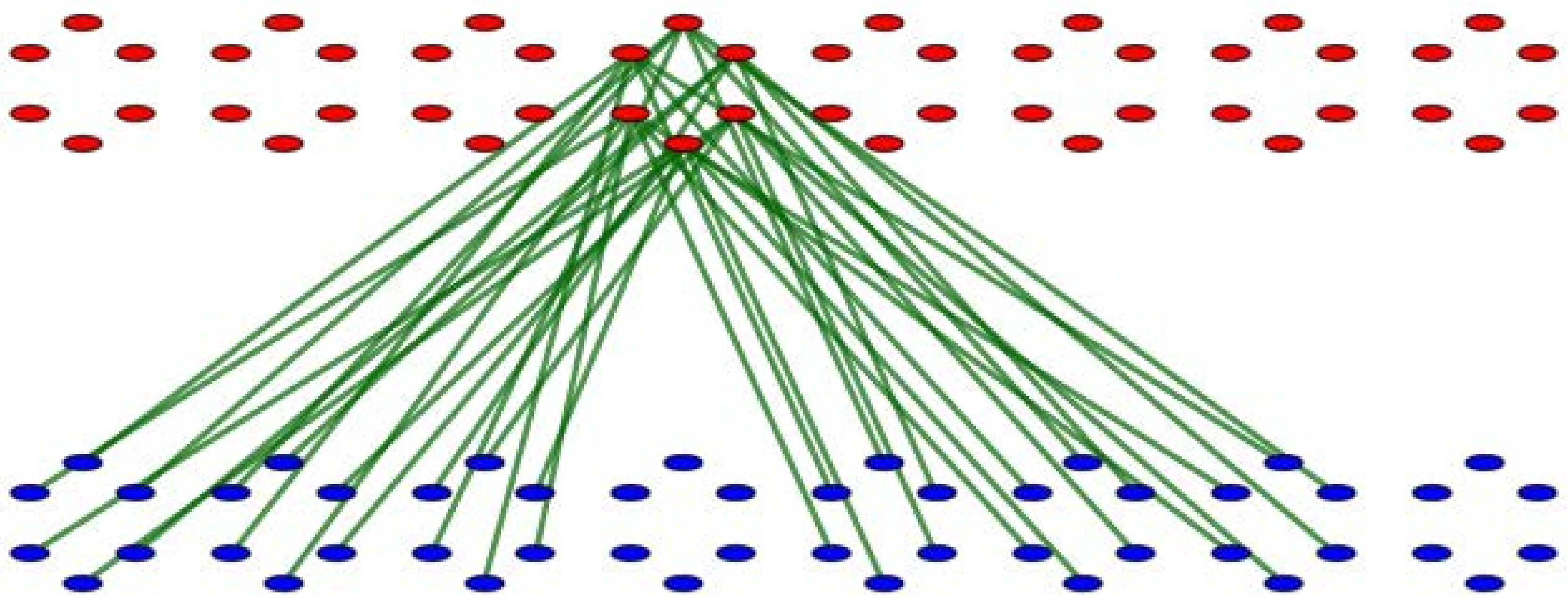
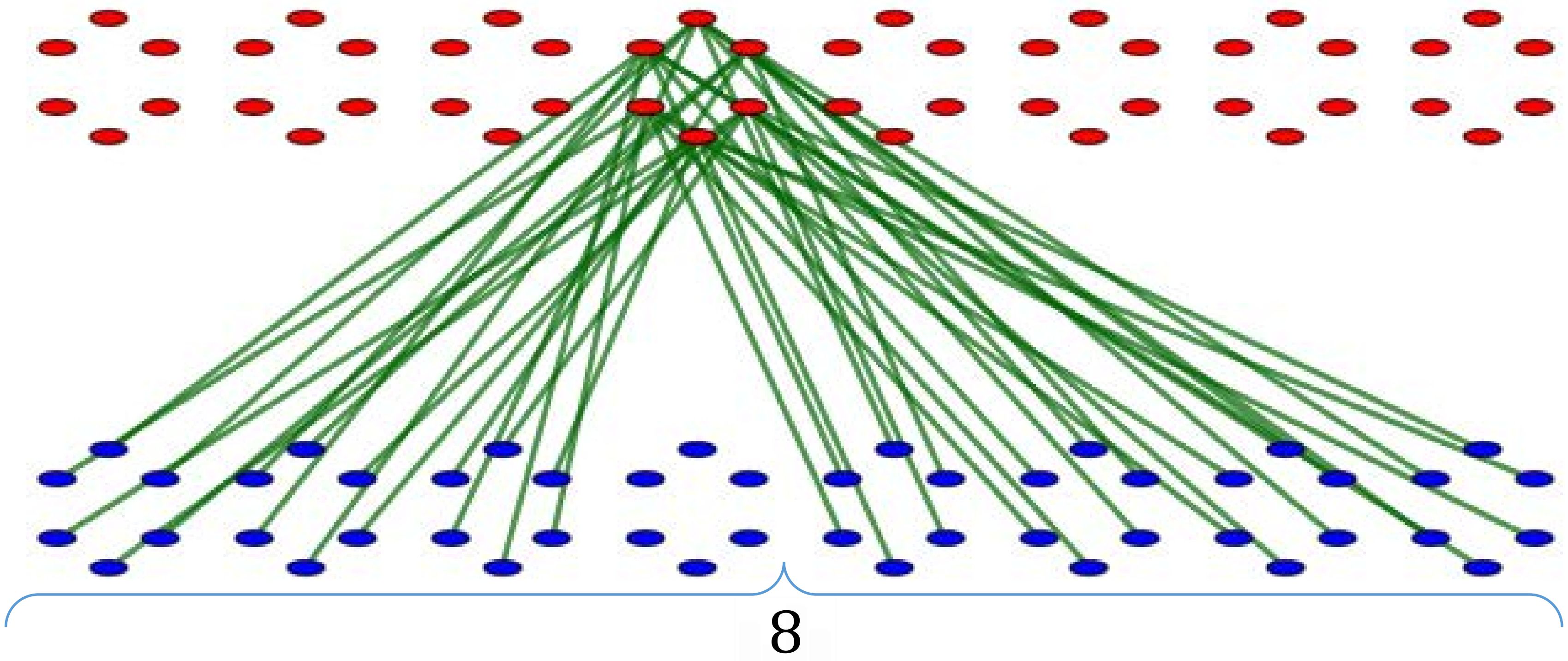
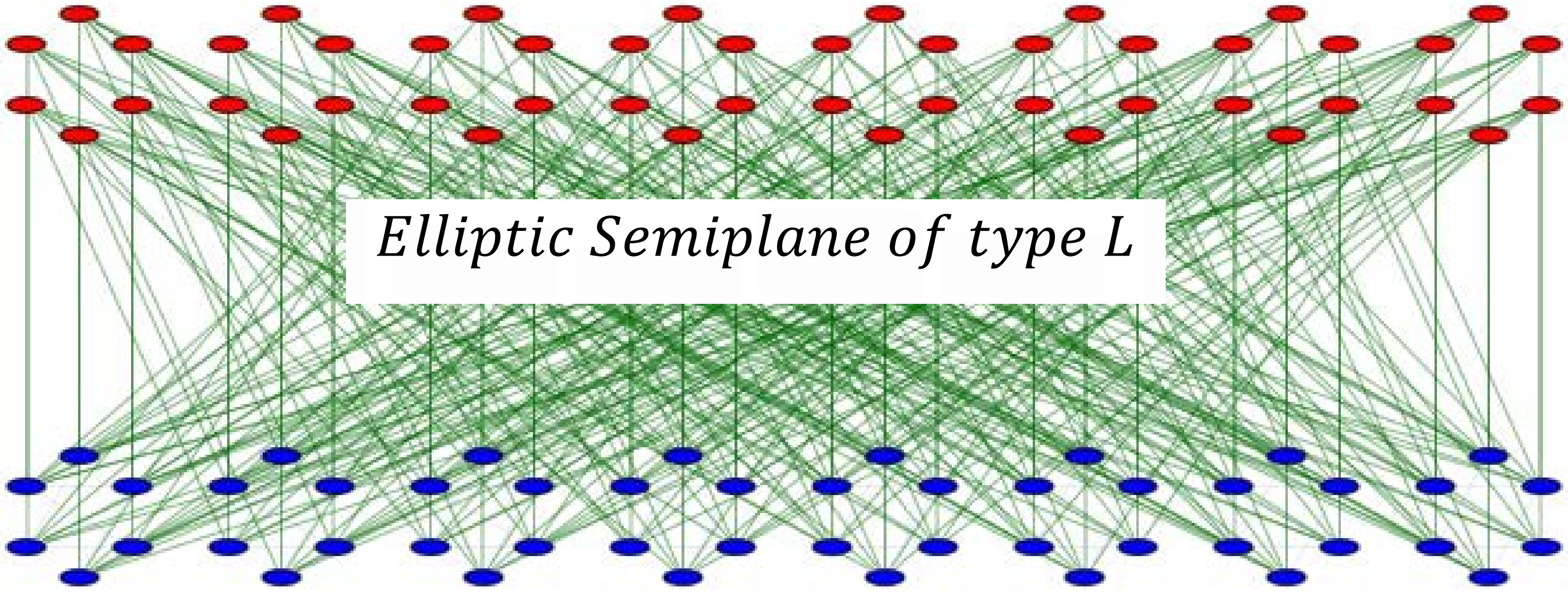


$$q = 7 \longrightarrow \begin{array}{c} 2\ (q+1)(q-1)\ \ vertices \\ 2 \cdot 8 \cdot 6 = 96\ vertices \end{array}$$



$q = 7 \longrightarrow$ $7-regular$ $girth\ 6$

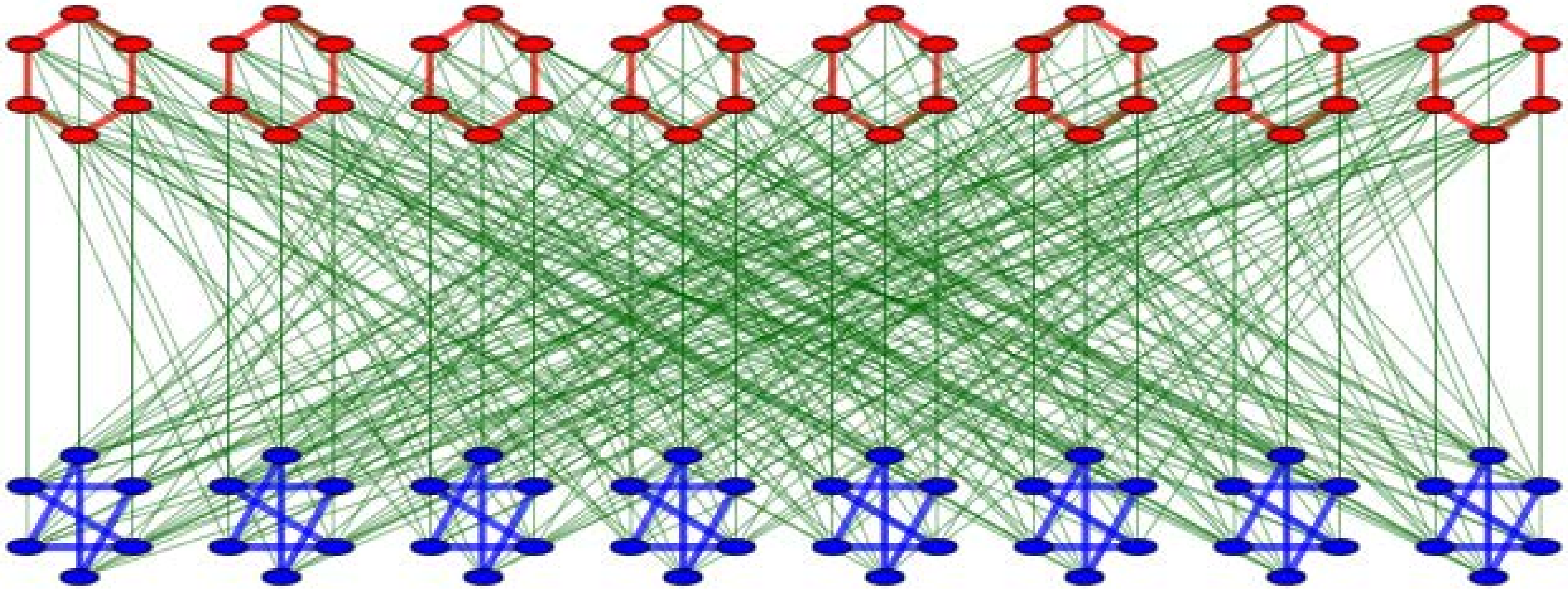
$2 \cdot 8 \cdot 6 = 96\ vertices$



Elliptic Semiplane of type L

$q = 7 \longrightarrow$ $9-regular$ $girth\ 5$

$2 \cdot 8 \cdot 6 = 96\ vertices$



Amalgamations

Let G_L, G_P, H_L, H_P be graphs such that $V(G_L), V(G_P), V(H_L), V(H_P) \subseteq \mathbb{F}_q^*$.

The amalgamation of these graphs into $\mathcal{L}_q$ is the graph denoted $\mathcal{L}_q(G_L, G_P, H_L, H_P)$ with vertex set $V_L \cup V_P$ and edge set $E_L \cup E_P \cup E$, where

- $V_L = \bigcup_{a \in \mathbb{F}_q} \{\ell[a,i] : i \in V(G_L)\} \cup \{\ell[q,i] : i \in V(H_L)\}$,
- $V_P = \bigcup_{x \in \mathbb{F}_q} \{p(x,j) : j \in V(G_P)\} \cup \{p(q,j) : j \in V(H_P)\}$,

and

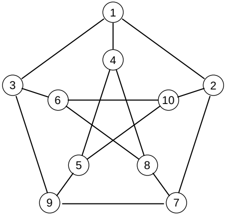
- $E_L = \left(\bigcup_{a \in \mathbb{F}_q} \{\{\ell[a,i], \ell[a,i']\} : ii' \in E(G_L)\}\right) \cup \{\{\ell[q,i], \ell[q,i']\} : ii' \in E(H_L)\}$,
- $E_P = \left(\bigcup_{x \in \mathbb{F}_q} \{\{p(x,j), p(x,j')\} : jj' \in E(G_P)\}\right) \cup \{\{p(q,j), p(q,j')\} : jj' \in E(H_P)\}$,
- $E = E(\mathcal{L}_q[V_L \cup V_P])$, where $\mathcal{L}_q[V_L \cup V_P]$ denotes the subgraph of $\mathcal{L}_q$ induced by $V_L \cup V_P$.

A difficult problem

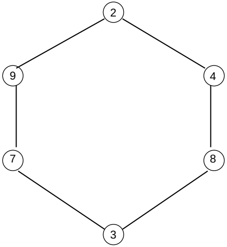
To construct suitable graphs G_L, G_P, H_L, H_P for obtaining $\mathcal{L}_q(G_L, G_P, H_L, H_P)$, a $q+t$-regular of girth 5.

For $q = 11$ to construct the $(13,5)$-graph with $rec(13,5) = 226$:

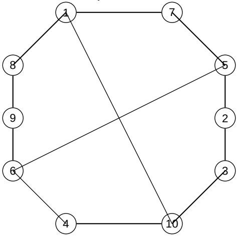
G_L



H_L



G_P

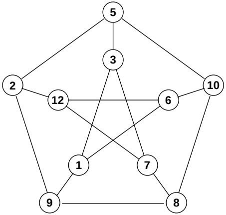


$H_P = \emptyset$

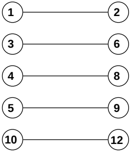
Take $q = 13$ to construct the $(14, 5)$-graph with $rec(14, 5) = 280$:

G_L

$S_L = \{4, 11\}$

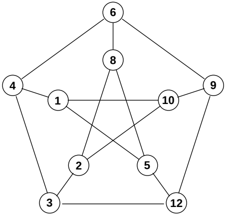


H_L

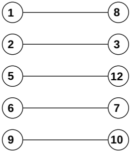


G_P

$S_P = \{7, 11\}$



H_P



Theorem [ABBs]

Let $S_L, S_P \subset \mathbb{F}_q^*$. Let G_L, G_P, H_L and H_P be graphs with $g \geq 5$ such that:

- $V(G_L) = \mathbb{F}_q^* \setminus S_L$ and $d(u) \in \{r + |S_P|, r + |S_P| + 1\}$, for $u \in V(G_L)$.
  Let $T_L = \{u \in V(G_L) : d(u) = r + |S_P| + 1\}$;
  $V(G_P) = \mathbb{F}_q^* \setminus S_P$ and $d(u) \in \{r + |S_L|, r + |S_L| + 1\}$ for $u \in V(G_P)$.
  Let $T_P = \{u \in V(G_P) : d(u) = r + |S_L| + 1\}$.
- H_L and H_P are r-regular graphs with $V(H_L) = \mathbb{F}_q^* \setminus (S_P \cup T_P)$ and $V(H_P) = \mathbb{F}_q^* \setminus (S_L \cup T_L)$.
- $E(H_L) \cap E(G_P) = \emptyset$, $E(H_P) \cap E(G_L) = \emptyset$ and G_L and G_P have disjoint sets of Cayley colors.

The graph $\mathcal{L}_q(G_L, G_P, H_L, H_P)$ is $(q+r)$-regular, has order $2(q^2-1) - (q+1)(|S_L| + |S_P|) - (|T_L| + |T_P|)$ and girth ≥ 5.

Cayley color of an edge

Let G be a graph with $V(G) \subseteq \mathbb{F}_q^*$. Let $yz \in E(G)$. The Cayley color of yz is denoted as $\omega(yz)$ and defined:

$$\omega(yz) = (yz^{-1})^{\pm 1}$$

Further new values of $rec(k,5)$ for $13 \leq k \leq 33$.

k	$rec(k,5)$	Due to	New $rec(k,5)$ ABBs
13	230	Exoo	*226*
14	284	AABL12	*280*
15	310	AABL12	
16	336	Jørgensen 2005	
17	436	AABB17	
18	468	AABB17	
19	500	AABB17	
20	564	AABB17	
21	666	AABB17	*658*
22	704	AABB17	
23	880	Funk2009	*874*
24	924	Funk2009	*920*
25	968	Funk2009	*960*
26	1012	Funk2009	*1010*
27	1056	Funk2009	*1054*
28	1200	Funk2009	*1192*
29	1248	Funk2009	*1240*
30	1404	Funk2009	*1392*
31	1456	Funk2009	*1444*
32	1624	AABB17	*1608*
33	1680	AABB17	*1664*

Theorem [ABBs]

Let $q \geq 61$ be an odd prime power and r_q the integer shown below. Then $n(k,5) \leq 2(q(k-r_q)-1)$.

$$r_q = \begin{cases} 7 & \text{if } q \in \{61, 71, \ldots, 89\} \\ 8 & \text{if } q \in \{101, \ldots, 109\} \\ 9 & \text{if } q \in \{97, 113, \ldots, 139\} \setminus \{127, 131\} \\ 10 & \text{if } q \in \{127, 131, 149, \ldots, 181\} \\ 11 & \text{if } q \in \{191, \ldots, 223\} \\ 12 & \text{if } q \in \{227, \ldots, 239\} \\ 13 & \text{if } q = 241 \\ 12 & \text{if } q > 241 \end{cases}$$

Theorem [ABBs]

Let $p \geq 13$ be a prime power such that $q = 2(p^2 + p + 1) + 1$ is also a prime power. Then

$$n\left(q + \frac{\sqrt{2q-5} - 1}{2}, 5\right) \leq 2q^2 - q - 1.$$

p	q=2(p^2+p+1)+1	k	rec(k,5)=2(q^2-q-1)	rec(k,5)=2(qk-12q-1)	k^2+3
13	367	380	269010	270110	144403
16	547	563	597870	602792	316972
25	1303	1328	3394314	3429494	1763587
31	1987	2018	7894350	7971842	4072327
49	4903	4952	48073914	48441638	24522307
91	16747	16838	560907270	563570042	283518247
139	38923	39062	3029960934	3039886298	1525839847
193	74887	75080	11216050650	11243234630	5637006403

Thanks!