Partially metric association schemes with a small multiplicity

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Joint work with Jack Koolen and Jongyook Park

Celebrating the work of....

Willem Haemers, Felix Lazebnik, and Andrew Woldar

Overview and intro

Association schemes are symmetric (except at the end of the talk)

We interpret relations of schemes as graphs (scheme graphs)

Every association scheme has an extremely small multiplicity (1)

Product constructions ? Multiplicity 2 ?

Focus on multiplicity 3. Which schemes are most interesting?

Partially metric schemes!

Tools: cosines, Godsil's bound, Terwilliger's light tail, Yamazaki's lemma

All partially metric schemes with a multiplicity 3

Time? Nonsymmetric schemes?

Association schemes

Let X be a finite set. An *association scheme* with rank $d+1$ on X is a pair $(X, \mathcal{R})$ such that

(i) $\mathcal{R} = \{R_0, R_1, \cdots, R_d\}$ is a partition of $X \times X$,

(ii) $R_0 := \{(x, x) \mid x \in X\}$,

(iii) $R_i = R_i^\top$ for each i, i.e., if $(x, y) \in R_i$ then $(y, x) \in R_i$,

(iv) there are numbers p_{ij}^h — the *intersection numbers* of $(X, \mathcal{R})$ — such that for every pair $(x, y) \in R_h$ the number of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^h.

Algebra

(i)' $\sum_{i=0}^{d} A_i = J$, where J is the all-one matrix,

(ii)' $A_0 = I$, where I is the identity matrix,

(iii)' $A_i^\top = A_i$ for all i,

(iv)' $A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h$.

The *Bose-Mesner algebra* $\mathcal{M} = \langle A_i \mid i = 0, \ldots, d \rangle$ has a basis of *minimal scheme idempotents* $E_0 = \frac{1}{n}J, E_1, \cdots, E_d$. The rank of E_j is denoted by m_j and is called the *multiplicity* of E_j, for $0 \le j \le d$.

$$A_i = \sum_{j=0}^{d} P_{ji} E_j \quad \text{and} \quad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

Geometry

$(E_j)_{xx} = \frac{Q_{0j}}{n} = \frac{m_j}{n}$ for all $x \in X$.

For $(x, y) \in R_i$, let $\omega_{xy} = \omega_{xy}(j) = \frac{(E_j)_{xy}}{(E_j)_{xx}} = \frac{Q_{ij}}{m_j}$. We call these numbers $\omega_i = \omega_i(j) = \frac{Q_{ij}}{m_j}$ the *cosines* corresponding to E_j, and note that $\omega_0 = 1$.

If E is a minimal idempotent with multiplicity m, then $E = UU^\top$, with U an $n \times m$ matrix with columns forming an orthonormal basis of the eigenspace of E for its eigenvalue 1.

For every vertex x we denote by $\hat{x}$ the row of U that corresponds to x, normalized to length 1. Now the inner product $\langle \hat{x}, \hat{y} \rangle$ is equal to $\frac{n}{m} E_{xy} = \omega_{xy}$.

Interesting schemes

A scheme is called *primitive* if all nontrivial relations are connected.

Bannai and Bannai 2006

The only primitive scheme with a multiplicity 3 is the scheme of the tetrahedron (K_4).

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The *direct product* of $(X, \mathcal{R})$ and $(X', \mathcal{R}')$ is the association scheme with relation matrices $A_i \otimes A'_j$ for $i = 0, 1, \ldots, d$ and $j = 0, 1, \ldots, d'$.

Starting from an association scheme $(X, \mathcal{R})$ with a multiplicity 3, one can construct other association schemes with a multiplicity 3 by taking the direct product of $(X, \mathcal{R})$ with any other scheme. Also other kinds of product constructions for association schemes are possible, giving rise to many association schemes with a multiplicity 3.

Distance-regular graphs

For a connected graph Γ with diameter D, the *distance-i graph* Γ_i of Γ ($0 \leq i \leq D$) is the graph whose vertices are those of Γ and whose edges are the pairs of vertices at mutual distance i in Γ.
A connected graph is called *distance-regular* if the distance-i graphs ($0 \leq i \leq D$) form an association scheme (a so-called *metric scheme*).

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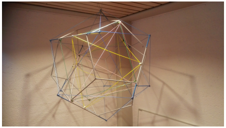
Godsil 80s

The distance-regular graphs with a multiplicity 3 are the graphs of the Platonic solids and the regular complete 4-partite graphs.



Distance-regular graphs with a multiplicity up to 8 have been classified (Zhu, Martin, Koolen, Godsil 90s).

Partially metric schemes

Yamazaki 1998

If Γ is a connected cubic scheme graph, then the distance-2 graph is also a relation of the scheme.

We call a scheme *partially metric* if it has a connected scheme graph whose distance-2 graph is also a relation of the scheme.

We adopt 'distance-regular graph' notation as far as possible: $a_1 = p^1_{11}$, etc.

Partially metric schemes with a multiplicity 3

Let $(X, \mathcal{R})$ be a partially metric scheme with rank $d+1$ and a multiplicity 3, and let Γ be the corresponding scheme graph. Then

(i) $d = 1$ and Γ is the tetrahedron (the complete graph on 4 vertices),

(ii) $d = 3$ and Γ is the cube,

(iii) $d = 5$ and Γ is the Möbius-Kantor graph,

(iv) $d = 6$ and Γ is the Nauru graph,

(v) $d = 11$ and Γ is the Foster graph F048A,

(vi) $d = 5$ and Γ is the dodecahedron,

(vii) $d = 11$ and Γ is the bipartite double of the dodecahedron,

(viii) $d = 3$ and Γ is the icosahedron,

(ix) $d = 2$ and Γ is the octahedron,

(x) $d = 2$ and Γ is a regular complete 4-partite graph.

Moreover, $(X, \mathcal{R})$ is uniquely determined by Γ. In all cases, except *(vii)*, this is the scheme that is generated by Γ. In case *(vii)*, the scheme is the *bipartite double scheme* of the scheme of case *(vi)*.

Godsil's bound

Godsil's bound (Cámara et al. 2013)

Let $(X, \mathcal{R})$ be a partially metric association scheme and assume that the corresponding scheme graph Γ has valency $k \geq 3$. Let E be a minimal scheme idempotent of $(X, \mathcal{R})$ with multiplicity m for corresponding eigenvalue $\theta \neq \pm k$. If Γ is not complete multipartite, then

$$k \leq \frac{(m+2)(m-1)}{2}.$$

$m = 1, 2$ are trivial.

$m = 3$ and complete multipartite implies regular complete 4-partite or the octahedron (cases (ix) and (x)).

$m = 3, k \leq 5$:?

$m = 3,\ k > 3$

$k > m$ implies $p^1_{11} = a_1 > 0$ because of local eigenvalues (à la Terwilliger).

$k = 5, a_1 = 2$ gives the icosahedron (case (viii)).

$k = 4, a_1 = 2$ gives the octahedron (case (ix)).

$k = 4, a_1 = 1$ implies ($m = 3$)-eigenvalue $\theta = -2$, which is excluded because of a light tail argument (à la Jurišić, Terwilliger, Žitnik).

$m = 3$, cubic graphs

$k = 3, a_1 > 0$ gives the tetrahedron (K_4; case (i)).

$k = 3, a_1 = 0, c_2 > 1$ gives the cube (case (ii)).

What remains: $k = 3, a_1 = 0, c_2 = 1$.

Γ is bipartite. If not, then consider *bipartite double scheme* (direct product with K_2 scheme).

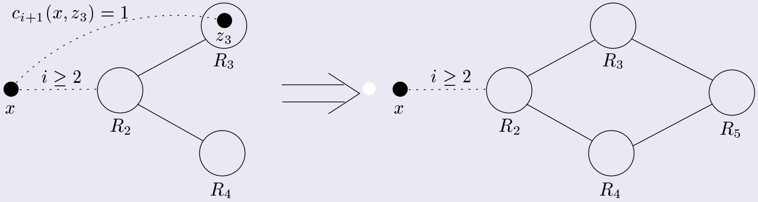
Tools: cosines

$\theta E = AE$ implies $\theta \omega_h = \sum_{\ell=1}^{d} p_{1\ell}^{h} \omega_\ell$

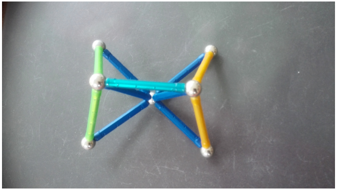
- $\omega_0 = 1$,
- $\theta \omega_0 = 3\omega_1$, so $\omega_1 = \theta/3$,
- $\theta \omega_1 = \omega_0 + 2\omega_2$, so $\omega_2 = \frac{1}{6}(\theta^2 - 3)$.

Tools: Yamazaki's lemma

Yamazaki 1998



Tools: a double fork



Let u_1 and u_2 be two adjacent vertices in Γ, let v_1, v_2 be the other two neighbors of u_1, and v_3, v_4 be the other two neighbors of u_2. Fix another vertex x, and let $\psi_i = \omega_{xu_i}$ $(i = 1, 2)$ and $\phi_i = \omega_{xv_i}$ $(i = 1, 2, 3, 4)$ be the respective cosines corresponding to E. Then

$$\phi_3, \phi_4 = \frac{1}{2}(\theta\psi_2 - \psi_1 \pm (\phi_1 - \phi_2)).$$

The relevant eigenvalues

- $\omega_0 = 1, \omega_1 = \frac{1}{3}\theta, \omega_2 = \frac{1}{6}(\theta^2 - 3)$
- $\omega_{3,4} = \frac{1}{2}(\theta\omega_2 - \omega_1 \pm \omega_0 \mp \omega_2) = \frac{1}{12}(\theta^3 \mp \theta^2 - 5\theta \pm 9)$
- $\omega_5 = \frac{1}{2}(\theta\omega_3 - \omega_2 + \omega_1 - \omega_4) = \frac{1}{24}(\theta^4 - 2\theta^3 - 8\theta^2 + 18\theta + 15)$
- $\omega_6 = \frac{1}{2}(\theta\omega_3 - \omega_2 - \omega_1 + \omega_4) = \frac{1}{24}(\theta^4 - 6\theta^2 - 3)$
- $\omega_7 = \frac{1}{2}(\theta\omega_4 - \omega_2 + \omega_1 - \omega_3) = \frac{1}{24}(\theta^4 - 6\theta^2 - 3)$
- $\omega_8 = \frac{1}{2}(\theta\omega_4 - \omega_2 - \omega_1 + \omega_3) = \frac{1}{24}(\theta^4 + 2\theta^3 - 8\theta^2 - 18\theta + 15)$

$\omega_i \neq \omega_2$ for $i = 5, 6, 7, 8$, hence $c_3 = 1$ if $\theta \neq \pm 1$

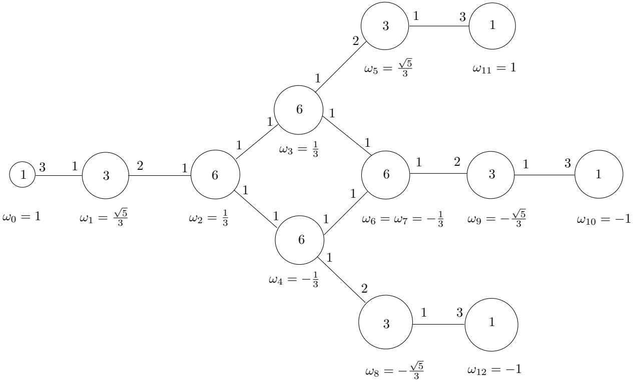
$c_3 = 1$

Yamazaki: $p^5_{14} \neq 0$ or $p^6_{14} \neq 0$

ω_9 and the double fork implies that $p^5_{14} = 0$ and

$\theta = \pm 1, \pm\sqrt{5}$

The dodecahedron and its bipartite double



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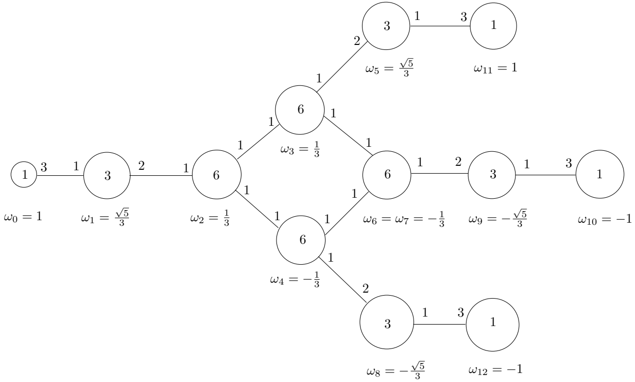
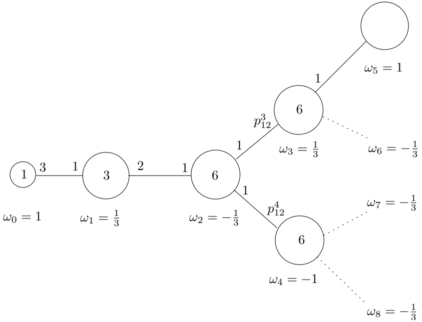
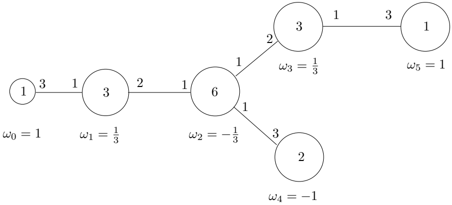


The dodecahedron has spectrum $\{3^1, \sqrt{5}^3, 1^5, 0^4, -2^4, -\sqrt{5}^3\}$.
Bipartite double has spectrum $\{3^1, \sqrt{5}^6, 2^4, 1^5, 0^8, -1^5, -2^4, -\sqrt{5}^6, -3^1\}$.

Partial relation-distribution diagram for $\theta = 1$



The Möbius-Kantor graph



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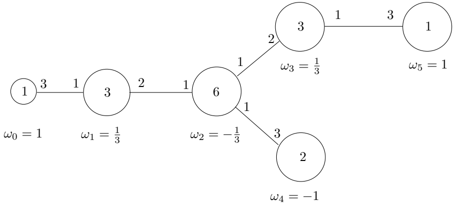
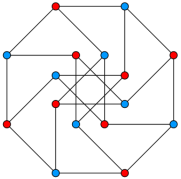
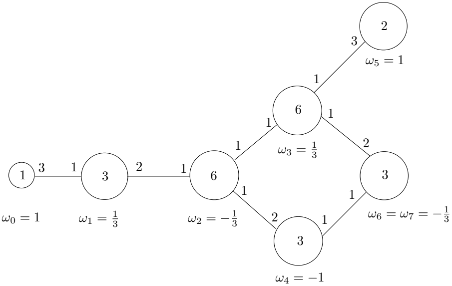




The Möbius-Kantor graph is the unique double cover of the cube without 4-cycles. It is isomorphic to the generalized Petersen graph $GP(8,3)$ and has spectrum $\{3^1, \sqrt{3}^4, 1^3, -1^3, -\sqrt{3}^4, -3^1\}$. It is 2-arc-transitive and also known as the Foster graph F016A.

The Nauru graph



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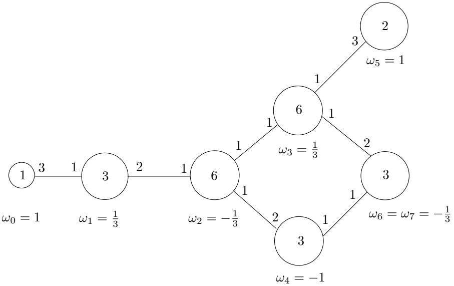
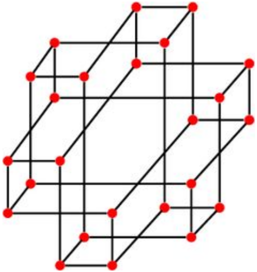
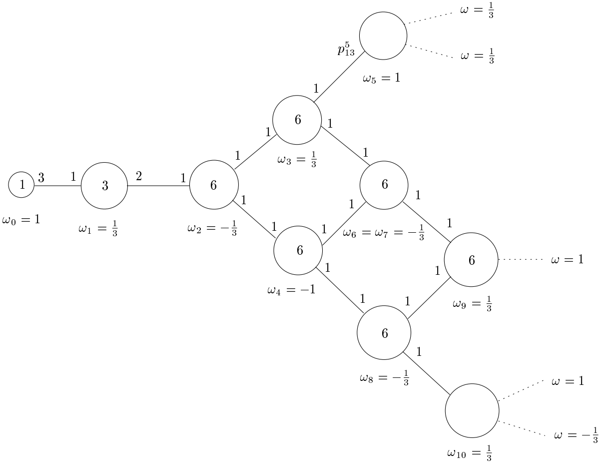




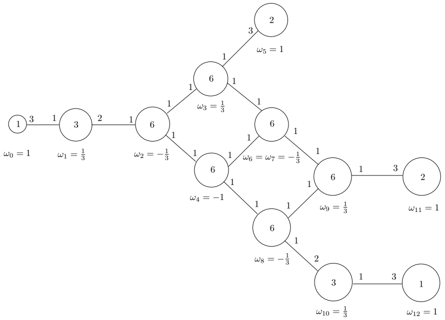
The Nauru graph is a triple cover of the cube. It is isomorphic to the generalized Petersen graph $GP(12,5)$ and has spectrum $\{3^1, 2^6, 1^3, 0^4, -1^3, -2^6, -3^1\}$. It is 2-arc-transitive and also known as the Foster graph F024A.

Girth 8



$\omega = \frac{1}{3}$
$\omega = \frac{1}{3}$
p^5_{13}
$\omega_5 = 1$
1
6
1
1
$\omega_3 = \frac{1}{3}$
1
1
3
1
3
2
1
6
1
6
1
$\omega_0 = 1$
$\omega_1 = \frac{1}{3}$
$\omega_2 = -\frac{1}{3}$
1
1
1
6
$\omega_6 = \omega_7 = -\frac{1}{3}$
1
6
$\omega = 1$
$\omega_4 = -1$
1
1
1
$\omega_9 = \frac{1}{3}$
6
1
$\omega_8 = -\frac{1}{3}$
$\omega = 1$
$\omega = -\frac{1}{3}$
$\omega_{10} = \frac{1}{3}$

The Foster graph F048A



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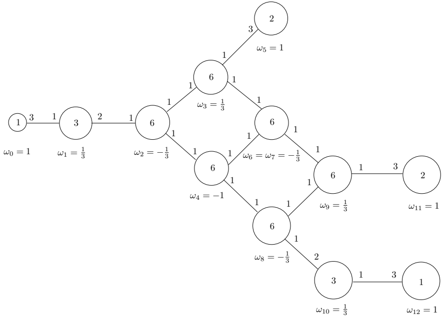
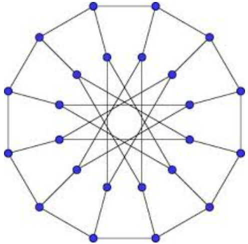
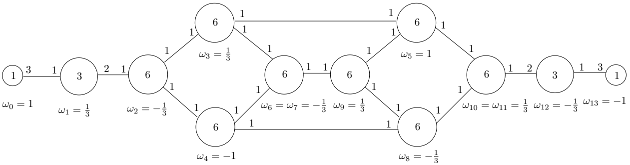
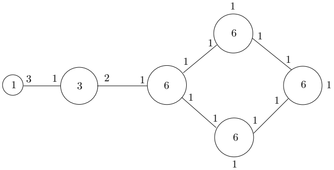




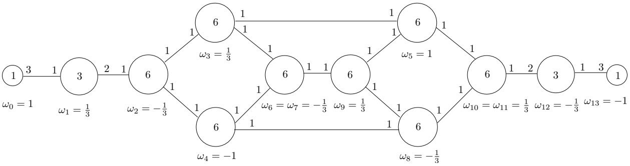
The Foster graph F048A is a 6-cover of the cube, a 3-cover of the Möbius-Kantor graph, and a 2-cover of the Nauru graph. It is isomorphic to the generalized Petersen graph $GP(24, 5)$ and has spectrum $\{3^1, \sqrt{6}^4, 2^6, \sqrt{3}^4, 1^3, 0^{12}, -1^3, -\sqrt{3}^4, -2^6, -\sqrt{6}^4, -3^1\}$.

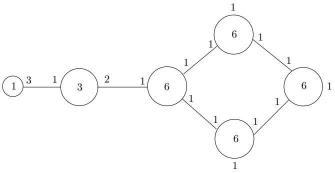
The Coxeter graph

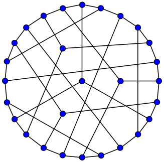




The Coxeter graph







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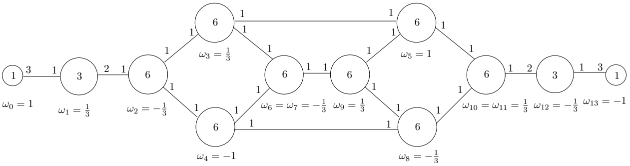
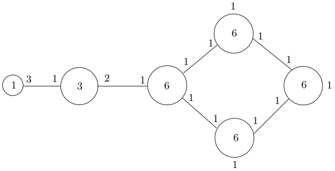
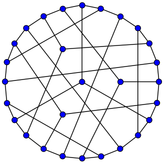
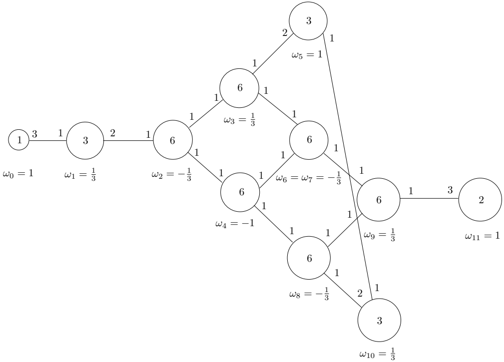






There are no such association schemes

Another 3-cover of the Möbius-Kantor graph



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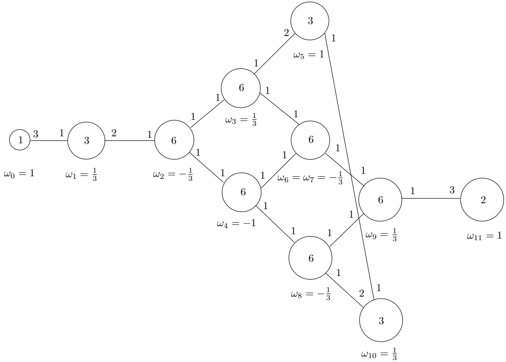

There is no such association scheme

Non-symmetric schemes

Feng, Kwak, and Wang's (2002, 2005) arc-transitive covers of the cube.

Let C be an order n cyclic permutation matrix and $N = \begin{bmatrix} I & I & I & 0 \\ I & C & 0 & I \\ I & 0 & C^{k+1} & C^k \\ 0 & I & C^k & C^k \end{bmatrix}$.

Let n and $k \leq n-1$ be such that k^2+k+1 is a multiple of n. Then the bipartite graph Γ with bipartite incidence matrix N is arc-transitive and it has eigenvalues ± 1 with multiplicity three.

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$(n,k)=(1,0)$ and $(n,k)=(3,1)$: cube and Nauru graph.
All other examples: non-symmetric (non-commutative) schemes.

An infinite family of non-commutative association schemes with a connected symmetric relation having an eigenvalue with multiplicity 3.

Recess