The smallest eigenvalues in the Hamming scheme

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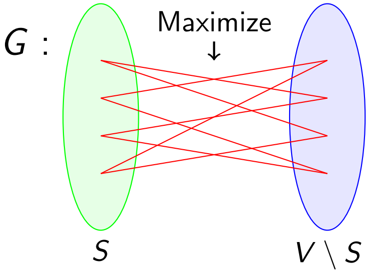
Max-Cut Problem

Max-cut problem (NP-Hard): Given $G = (V, E)$ determine $\max_{S \subset V} e(S, V \setminus S)$.



Approximation methods

- **Goemans-Williamson** (1995): 0.878 approximation algorithm using SDP techniques.
- **Van Dam and Sotirov** (2016): SDP relaxation of max-k-cut

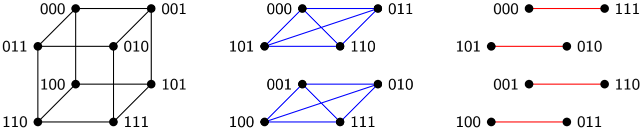
$$\begin{aligned} \max \quad & \frac{1}{2} tr(LY) \\ s.t. \quad & \text{diag}(Y) = j_n \\ & kY - J_n \succeq 0, \; Y \geq 0 \end{aligned} \tag{1}$$

- **Van Dam and Sotirov** (2016): Eigenvalue bound for max-k-cut

$$\frac{n(k-1)}{2k} \lambda_{\max}(L) \tag{2}$$

Graphs in the Hamming scheme

$H(d,q,j)$ has $V = Q^d$, $|Q| = q$ with $x \sim y \iff d(x,y) = j$.



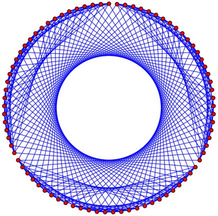
$H(3,2,1)$ $H(3,2,2)$ $H(3,2,3)$

Eigenvalues of $H(d,q,j)$

The eigenvalues of the graph $H(d,q,j)$ are given by the Krawtchouk polynomial

$$K_j(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h} \binom{d-i}{j-h} \text{ for } 0 \leq i \leq d.$$

Ex. $H(4,3,1)$



$$P = \begin{bmatrix} 1 & 8 & 24 & 32 & 16 \\ 1 & 5 & 6 & -4 & -8 \\ 1 & 2 & -3 & -4 & 4 \\ 1 & -1 & -3 & 5 & -2 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

Van Dam and Sotirov conjecture

Conjecture 1 (Van Dam and Sotirov (2016))

Let $q \geq 2$ and $j \geq d - \frac{d-1}{q}$, with j even if $q = 2$. Then

$$K_j(1) = \min_{0 \leq i \leq d} K_j(i).$$

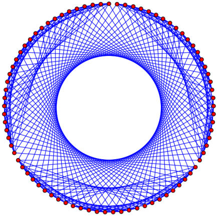
- **Alon and Sudakov** (2000): For $q = 2$, $K_j(1) \leq K_j(i)$ for j even $j > d/2$ when d is large enough with j/d fixed.
- **Dumer and Kapralova** (2013): For $q = 2$, $|K_j(1)| > |K_j(i)|$ for $1 \leq j \leq d-1$ and $1 \leq i \leq d-1$ unless $d = 2j$, for which the maximum occurs at $i = 2$.

Our result

Theorem 1 (Brouwer, Cioabă, Ihringer and McGinnis (2017))

Let $q \geq 3$ *and* $d - \frac{d-1}{q} \leq j \leq d$.
(i) $K_j(1) \leq K_j(i)$ *for all* i, $0 \leq i \leq d$.
(ii) $|K_j(i)| \leq |K_j(1)|$ *for all* $i \geq 1$, *unless* $(d, q, j) = (4, 3, 3)$.

Ex. $H(4, 3, 1)$



$H(4, 3, 3)$ ↓

$$P = \begin{bmatrix} 1 & 8 & 24 & 32 & 16 \\ 1 & 5 & 6 & -4 & -8 \\ 1 & 2 & -3 & -4 & 4 \\ 1 & -1 & -3 & 5 & -2 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

Proof outline

Recurrence relation

$$(q-1)(d-i)K_j(i+1)-(i+(q-1)(d-i)-qj)K_j(i)+iK_j(i-1)=0$$

Lemma 2

Let $1 < i < d$ and $d - \frac{d-1}{q} \leq j \leq d$. If $qj \leq 2(q-1)(d-i)$, then $|K_j(i+1)| \leq \max(|K_j(i-1)|,\ |K_j(i)|)$.

If $qj \leq 2(q-1)(d-i+1)$, applying Lemma 2 and induction on i yields

$$|K_j(i)| \leq \max\left(|K_j(1)|, |K_j(2)|\right).$$

Proof outline

$$K_j(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h} \binom{d-i}{j-h} \text{ for } 0 \leq i \leq d$$

Lemma 3

$|K_j(i)| \leq \sum_{h \geq i+j-d} (q-1)^{j-h} \binom{i}{h} \binom{d-i}{j-h} \leq (q-1)^{d-i} \binom{d}{j}$.

Lemma 3 implies $|K_j(i)| \leq |K_j(1)|$ when $d \leq (q-1)^{i+j-d-1}(qj - (q-1)d)$.

Assuming $qj > 2(q-1)(d-i+1)$ this inequality holds for $d \geq 30$.

Johnson scheme

$J(n, d, j)$ has $V = \binom{[n]}{d}$ with $x \sim y \iff |x \cap y| = d - j$. The eigenvalues are given by the Eberlein polynomial

$$E_j(i) = \sum_{h=0}^{j} (-1)^h \binom{i}{h} \binom{d-i}{j-h} \binom{n-d-i}{j-h} \text{ for } 0 \leq i \leq d.$$

Conjecture 2 (Karloff (1999))

If $j > d/2$, then the smallest eigenvalue of $J(2d, d, j)$ and second largest in absolute value is $E_j(1)$.

Proof outline

Recurrence relation

For $i, j \geq 1$, $E_j^{2(d+1),d+1}(i) = E_j^{2d,d}(i-1) - E_{j-1}^{2d,d}(i-1)$.

Lemma 4

Let $(j-1)(2d+1) \geq d^2$. *Then*
$E_j(0) + |E_{j-1}(1)| + |E_j(1)| \leq E_{j-1}(0)$.

Induction on d. Showing $|E_j^{2(d+1),d+1}(i)| \leq |E_j^{2(d+1),d+1}(1)|$ is equivalent to showing $|E_j(i-1) - E_{j-1}(i-1)| \leq |E_j(0) - E_{j-1}(0)|$. It suffices to show $|E_j(1)| + |E_{j-1}(1)| + E_j(0) \leq E_{j-1}(0)$.

Current work

- **Van Dam and Sotirov conjecture**: Proven.
- **Karloff conjecture**: Proven.
- Grassmanian, Dual polar, Bilinear forms and Alternating forms schemes