Toughness, connectivity and the spectrum of regular graphs

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Outline

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1. Toughness
2. Generalized connectivity
3. Spanning tree with bounded maximum degree

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Matrix and Eigenvalue

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- Let $\lambda = \max\{|\lambda_2|, |\lambda_3|, \cdots, |\lambda_n|\} = \max\{|\lambda_2|, |\lambda_n|\}$.

Toughness

- The toughness $t(G)$ of a connected graph G is defined as $t(G) = \min\{\frac{|S|}{c(G-S)}\}$, where the minimum is taken over all proper subset $S \subset V(G)$ such that $c(G-S) > 1$.

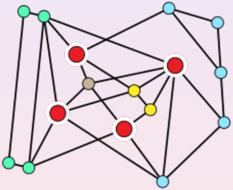

Figure: toughness $= 1$

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- Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness greater than t_0 is Hamiltonian.

Some results

- Theorem (Alon 1995)
  For any connected d-regular graph G, $t(G) > \frac{1}{3}(\frac{d^2}{d\lambda+\lambda^2} - 1)$.

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More results

- Theorem (Cioabă and G. 2016)
  For any connected d-regular graph G with $d \geq 3$ and edge connectivity $\kappa' < d$, $t(G) > \frac{d}{\lambda_2} - 1 \geq \frac{d}{\lambda} - 1$.

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- brief idea:
  1. Let G be a connected d-regular graph with edge connectivity κ'. Then $t(G) \geq \kappa'/d$.
  2. Let G be a d-regular graph with $d \geq 2$ and edge connectivity $\kappa' < d$. Then $\lambda_2(G) \geq d - \frac{2\kappa'}{d+1}$.

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- Brouwer's conjecture remains unsolved for the case $\kappa' = d$.

More results

- Theorem (Liu and Chen 2010)
  For any connected d-regular graph G, if

$$\lambda_2(G) < \begin{cases} d-1+\frac{3}{d+1}, & \text{if } d \text{ is even,} \\ d-1+\frac{2}{d+1}, & \text{if } d \text{ is odd,} \end{cases}$$

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- **Theorem** (Cioabă and Wong 2014)
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- Theorem (Cioabă and G. 2016)
  Let G be a connected d-regular graph with $d \geq 3$ and edge connectivity κ'. If $\kappa' = d$, or, if $\kappa' < d$ and

$$\lambda_{\lceil \frac{d}{d-\kappa'} \rceil}(G) < \begin{cases} \frac{d-2+\sqrt{d^2+12}}{2}, & \text{if } d \text{ is even,} \\ \frac{d-2+\sqrt{d^2+8}}{2}, & \text{if } d \text{ is odd,} \end{cases}$$

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  then $t(G) \geq 1$.
- Theorem (Cioabă and G. 2016)
  For any bipartite connected d-regular graph G with $\kappa' < d$, if $\lambda_{\lceil \frac{d}{d-\kappa'} \rceil}(G) < d - \frac{d-1}{2d}$, then $t(G) = 1$.

Useful tools: Interlacing Theorem

- Theorem
  Let A be a real symmetric $n \times n$ matrix and B be a principal $m \times m$ submatrix of A. Then $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A)$ for $1 \leq i \leq m$.

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- **Corollary**
  Let $S_1, S_2, \cdots, S_k$ be disjoint subsets of $V(G)$ with $e(S_i, S_j) = 0$ for $i \neq j$. Then

$$\lambda_k(G) \geq \lambda_k(G[\cup_{i=1}^k S_i]) \geq \min_{1 \leq i \leq k} \{\lambda_1(G[S_i])\}.$$

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- The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices of G whose removal produces a disconnected graph or a single vertex.

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- Given an integer $l \geq 2$, Chartrand, Kapoor, Lesniak and Lick defined the l-connectivity $\kappa_l(G)$ of a graph G to be the minimum number of vertices of G whose removal produces a disconnected graph with at least l components or a graph with fewer than l vertices.

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- By definition, for a noncomplete connected graph G, we have $t(G) = \min_{2 \leq l \leq \alpha}\{\frac{\kappa_l(G)}{l}\}$ where α is the independence number of G.

Results

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  For a d-regular graph, $\kappa \geq d - \lambda_2$.
- Theorem (Krivelevich and Sudakov 2006)
  For a d-regular graph, $\kappa \geq d - \frac{36\lambda^2}{d}$.
- Theorem (Cioabă and G. 2016)
  Let l, k be integers with $l \geq k \geq 2$. For any connected d-regular graph G with $|V(G)| \geq k + l - 1$, $d \geq 3$ and edge connectivity κ', if $\kappa' = d$, or, if $\kappa' < d$ and

$$\lambda_{\lceil \frac{(l-k+1)d}{d-\kappa'} \rceil}(G) < \begin{cases} \frac{d-2+\sqrt{d^2+12}}{2}, & \text{if } d \text{ is even,} \\ \frac{d-2+\sqrt{d^2+8}}{2}, & \text{if } d \text{ is odd,} \end{cases}$$

  then $\kappa_l(G) \geq k$.

Corollaries

- Corollary (Cioabă and G. 2016)
  Let $l \geq 2$. For any connected d-regular graph G with $|V(G)| \geq l+1$ and $d \geq 3$, if

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- Corollary (Cioabă and G. 2016)
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Spanning tree with bounded maximum degree

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- Theorem (Win 1989)
  Let $k \geq 2$ and G be a connected graph. If for any $S \subseteq V(G)$, $c(G-S) \leq (k-2)|S|+2$, then G has a spanning k-tree.

Spanning tree with bounded maximum degree

- Theorem (Wong 2013)
  Let $k \geq 3$ and G be a connected d-regular graph. If $\lambda_4 < d - \frac{d}{(k-2)(d+1)}$, then G has a spanning k-tree.

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- Theorem (Wong 2013)
  Let $k \geq 3$ and G be a connected d-regular graph. If $\lambda_4 < d - \frac{d}{(k-2)(d+1)}$, then G has a spanning k-tree.
- Theorem (Cioabă and G. 2016)
  Let $k \geq 3$ and G be a connected d-regular graph with edge connectivity κ'. Let $l = d - (k-2)\kappa'$. Each of the following statements holds.
  (i) If $l \leq 0$, then G has a spanning k-tree.
  (ii) If $l > 0$ and $\lambda_{\lceil \frac{3d}{l} \rceil} < d - \frac{d}{(k-2)(d+1)}$, then G has a spanning k-tree.

Hamiltonian graphs

- Conjecture (Krivelevich and Sudakov, 2002)
  Let G be a d-regular graph with n vertics and with the second largest absolute value λ. There exist a positive consitant C such that for large enough n, if $d/\lambda > C$, then G is Hamiltonian.

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- Recall: Theorem (Brouwer, 1995)
  For any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - 2$.
- Krivelevich and Sudakov proved, if $d/\lambda > f(n)$, then G is Hamiltonian.

Thank You