Al algebraic approach to lifts of digraphs

(joint work with C. Dalfó, M. Miller, J. Ryan, and J. Širáň)

Algebraic and Extremal Graph Theory
(A conference in honor of W. Haemers, F. Lazebnik, and A. Woldar)

University of Delaware, August 7–10, 2017

Outlook

1. Introduction

- $\Gamma = (V, E)$: Strongly connected digraph on n vertices. It can have loops and multiple arcs.
- Spectrum of the adjacency matrix A of Γ:

$$\operatorname{sp}\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\},$$

where λ_i and m_i are the roots of the characteristic polynomial and their multiplicities.

3. Voltage assignment and lifted digraphs

- Γ: Digraph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$.

3. Voltage assignment and lifted digraphs

- Γ: Digraph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$.
- Voltage assignment: It takes a base digraph and a group to obtain a new and larger digraph.
  Given a digraph Γ, and a finite group G with generating set Δ, a voltage assignment α is a mapping $\alpha : E \to \Delta$, that is, a labelling of the arcs with elements of G.

3. Voltage assignment and lifted digraphs

- Γ: Digraph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$.
- Voltage assignment: It takes a base digraph and a group to obtain a new and larger digraph.
  Given a digraph Γ, and a finite group G with generating set Δ, a voltage assignment α is a mapping $\alpha : E \to \Delta$, that is, a labelling of the arcs with elements of G.
- Lifted digraph Γ^α: Digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times \Delta$, where there is an arc from vertex (u, g) to vertex $(v, g\alpha(uv))$ if and only if $uv \in E$:

$$(u, g) \longrightarrow (v, g\alpha(uv)).$$

3. Voltage assignment and lifted digraphs

- Γ: Digraph with vertex set $V = V(\Gamma)$ and arc set $E = E(\Gamma)$.

- Voltage assignment: It takes a base digraph and a group to obtain a new and larger digraph.
  Given a digraph Γ, and a finite group G with generating set Δ, a voltage assignment α is a mapping $\alpha : E \to \Delta$, that is, a labelling of the arcs with elements of G.

- Lifted digraph Γ^α: Digraph with vertex set $V(\Gamma^\alpha) = V \times G$ and arc set $E(\Gamma^\alpha) = E \times \Delta$, where there is an arc from vertex (u, g) to vertex $(v, g\alpha(uv))$ if and only if $uv \in E$:

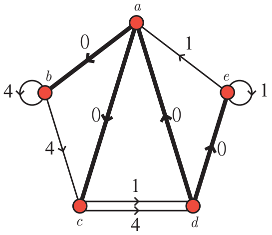
$$(u, g) \longrightarrow (v, g\alpha(uv)).$$

- In particular, the Cayley digraph $\mathrm{Cay}(\Gamma, \Delta)$ with $\Delta = \{g_1, \ldots, g_r\}$ can be seen as the lifted digraph Γ^α, where $\Gamma = K_1^r$ (a singleton with $V = \{u\}$ and $E = \{e_1, \ldots, e_r\}$ are r loops) and voltage assignment

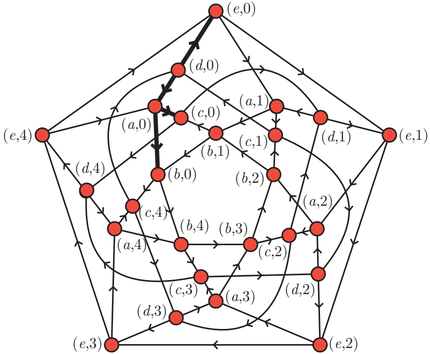
$$\begin{aligned} \alpha : E &\longrightarrow \Delta \\ e_i &\longrightarrow \alpha(e_i) = g_i \end{aligned}$$

3. Example: The Alegre digraph

- The Alegre digraph is the 2-regular digraph with $n = 25$ vertices and diameter 4.
  It was found by F., Yebra, and Alegre in 1984.
  It can be seen as the lifted digraph Γ^{α} of the base digraph Γ with voltage assignment α, group $G = \mathbb{Z}_5$ and $\Delta = \{0, 1, 4\} = \{0, \pm 1\}$.

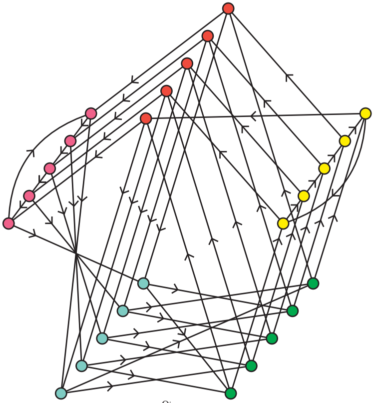


Γ



Γ^{α}

The Alegre digraph again and its spectrum



Γ^{α}

$$\operatorname{sp}\Gamma^{\alpha} = \left\{2, 0^{(10)}, i^{(5)}, -i^{(5)}, \tfrac{1}{2}(-1+\sqrt{5})^{(2)}, \tfrac{1}{2}(-1-\sqrt{5})^{(2)}\right\}$$

4. Matrix representation of a lifted digraph (cyclic group)

- Representation of a lifted digraph with a matrix whose size equals the order of the base graph when the group G of the voltage assignments is cyclic.

4. Matrix representation of a lifted digraph (cyclic group)

- Representation of a lifted digraph with a matrix whose size equals the order of the base graph when the group G of the voltage assignments is cyclic.
- Voltage assignment α: On the cyclic group $G = \mathbb{Z}_m = \{0, 1, \ldots, m-1\}$.

4. Matrix representation of a lifted digraph (cyclic group)

- Representation of a lifted digraph with a matrix whose size equals the order of the base graph when the group G of the voltage assignments is cyclic.

- Voltage assignment α: On the cyclic group $G = \mathbb{Z}_m = \{0, 1, \ldots, m-1\}$.

- Polynomial matrix $\boldsymbol{B}(z)$: A square matrix indexed by the vertices of Γ, and whose elements are polynomials in the quotient ring $\mathbb{R}_{m-1}[z] = \mathbb{R}[z]/(z^m)$, where (z^m) is the ideal generated by the polynomial z^m. Each entry of $\boldsymbol{B}(z)$ is represented by a polynomial

$$(\boldsymbol{B}(z))_{uv} = p_{uv}(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_{m-1} z^{m-1},$$

where

$$\alpha_i = \begin{cases} 1, & \text{if } uv \in E \text{ and } \alpha(uv) = i, \\ 0, & \text{otherwise.} \end{cases}$$

for $i = 0, \ldots, m-1$.

4. Matrix representation of a lifted digraph (cyclic group)

- The powers of $\boldsymbol{B}(z)$ give the same information as the powers of the adjacency matrix of the lifted digraph Γ^{α}.

4. Matrix representation of a lifted digraph (cyclic group)

- The powers of $\boldsymbol{B}(z)$ give the same information as the powers of the adjacency matrix of the lifted digraph Γ^{α}.
- **Lemma 4.** Let $(\boldsymbol{B}(z)^{\ell})_{uv} = \beta_0 + \beta_1 x + \cdots + \beta_{m-1} z^{m-1}$. Then, for every $i = 0, \ldots, m-1$, the coefficient β_i equals the number of walks of length ℓ in the lifted digraph Γ^{α}, from vertex (u, h) to vertex $(v, h+i)$ for every $h \in G$.

4. Matrix representation of a lifted digraph (cyclic group)

- The powers of $\boldsymbol{B}(z)$ give the same information as the powers of the adjacency matrix of the lifted digraph Γ^α.
- **Lemma 4.** Let $(\boldsymbol{B}(z)^\ell)_{uv} = \beta_0 + \beta_1 x + \cdots + \beta_{m-1} z^{m-1}$. Then, for every $i = 0, \ldots, m-1$, the coefficient β_i equals the number of walks of length ℓ in the lifted digraph Γ^α, from vertex (u, h) to vertex $(v, h+i)$ for every $h \in G$.
- The products of the entries (polynomials) of $\boldsymbol{B}(z)$ are in the ring $\mathbb{R}[z]/(z^m)$.

4. Example. The Alegre digraph

- Matrix representation:

$$\boldsymbol{B}(z) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & z^4 & z^4 & 0 & 0 \\ 0 & 0 & 0 & z+z^4 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ z & 0 & 0 & 0 & z \end{pmatrix}, \quad \boldsymbol{B}(1) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

- All coefficients α_i, for $i = 0, \ldots, 4$, of the polynomials of $\boldsymbol{I} + \boldsymbol{B}(z) + \boldsymbol{B}(z)^2 + \boldsymbol{B}(z)^3 + \boldsymbol{B}(z)^4$ are non-zero, since Γ^α has diameter four.

4. Example. The Alegre digraph

- Matrix representation:

$$\boldsymbol{B}(z) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & z^4 & z^4 & 0 & 0 \\ 0 & 0 & 0 & z+z^4 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ z & 0 & 0 & 0 & z \end{pmatrix}, \quad \boldsymbol{B}(1) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

- All coefficients α_i, for $i = 0, \ldots, 4$, of the polynomials of $\boldsymbol{I} + \boldsymbol{B}(z) + \boldsymbol{B}(z)^2 + \boldsymbol{B}(z)^3 + \boldsymbol{B}(z)^4$ are non-zero, since Γ^α has diameter four.

- For example, the entries of the first row of $\boldsymbol{I} + \boldsymbol{B}(z) + \boldsymbol{B}(z)^2 + \boldsymbol{B}(z)^3 + \boldsymbol{B}(z)^4$ are:
  $3 + z + z^2 + z^3 + z^4$, $1 + z + z^2 + z^3 + 2z^4$, $1 + z + z^2 + z^3 + 2z^4$,
  $1 + z + z^2 + z^3 + 2z^4$, $2 + z + z^2 + z^3 + z^4$.

The spectrum of the lifted digraph (cyclic group)

The spectrum of the lift Γ^α can be completely determined from the spectrum of the polynomial matrix $\boldsymbol{B}(z)$.

Proposition

Let $\Gamma = (V, E)$ be a base digraph on r vertices, with a voltage assignment α in $\mathbb{Z}_k$. Let $P(\lambda, z) = \det(\lambda I - \boldsymbol{B}(z))$ be the characteristic polynomial of the polynomial matrix $\boldsymbol{B}(z)$ of the voltage digraph (Γ, α). For $j = 0, \ldots, k-1$, let ω_j be the distinct k-th complex roots of unity. Then, the spectrum of the lift Γ^α is the multiset of kr roots λ of the k polynomials $P(\lambda, \omega_j)$ of degree r each, where $0 \leq j \leq k-1$; formally,

$$\operatorname{sp}\Gamma^\alpha = \{\lambda_{i,j} \,:\, P(\lambda_{i,j}, \omega_j) = 0,\ 1 \leq i \leq r,\ 0 \leq j \leq k-1\}.$$

The spectrum of the of the Alegre digraph

The polynomial matrix $\boldsymbol{B}(z)$ of the Alegre digraph has spectrum $\operatorname{sp}\boldsymbol{B} = \{0^{(2)}, i^{(1)}, -i^{(1)}, (z+\frac{1}{z})^{(1)}\}$.
Then, evaluating them at the 5-th roots of unity $\omega_i = e^{i\frac{2\pi}{5}}$, for $i = 0,1,2,3,4$, we get:

$z\backslash\lambda(z)$	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$(z+\frac{1}{z})^{(1)}$
1	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$2^{(1)}$
ω	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1+\sqrt{5})^{(1)}$
ω^2	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1-\sqrt{5})^{(1)}$
ω^3	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1+\sqrt{5})^{(1)}$
ω^4	$0^{(2)}$	$i^{(1)}$	$-i^{(1)}$	$\frac{1}{2}(-1-\sqrt{5})^{(1)}$

Table: The eigenvalues of the Alegre digraph.

The case of a general group

Let $\Gamma = (V, E)$ be a digraph with voltage assignment α on the group G. Its associated matrix $\boldsymbol{B}$ is a square matrix indexed by the vertices of Γ, and whose entries are elements of the group algebra $\mathbb{C}[G]$. Namely,

$$(\boldsymbol{B})_{uv} = \sum_{g \in G} \alpha_g g$$

where

$$\alpha_i = \begin{cases} 1 & \text{if } uv \in E \text{ and } \alpha(uv) = g, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, n$.

The number of walks

Lemma

Let

$$(\boldsymbol{B}^{\ell})_{uv} = \sum_{g \in G} a_g^{(\ell)} g.$$

Then, for every $g, h \in G$, the coefficient $a_g^{(\ell)}$ equals the number of walks of length ℓ in the lifted digraph Γ^{α}, from vertex (u, h) to vertex (v, hg). In particular, if $u = v$ and ι denotes the identity element of G, $a_{\iota}^{(\ell)}$ is the number of walks of length ℓ rooted at every vertex (u, g), for $g \in G$, of the lift.

The spectrum

Theorem

Let $\Gamma = (V, E)$ be a base digraph on r vertices, with a voltage assignment α in a group G with $|G| = n$. Assume that G has ν conjugacy classes with dimensions $d_1, \ldots, d_\nu$ (so, $\sum_{i=1}^{\nu} d_i^2 = n$). Let $\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_\nu$ be the irreducible representations of the group G. Let $\boldsymbol{\rho}_i(\boldsymbol{B})$ the complex matrix obtained from $\boldsymbol{B}$ by replacing each $g \in G$ by the $d_i \times d_i$ matrix $\boldsymbol{\rho}_i(g)$, and let $\mu_{u,j}$, $u \in V$, $j \in [1, d_i]$ denote its eigenvalues.
Then, the rn eigenvalues of the lift Γ^α are the rd_i eigenvalues of $\boldsymbol{\rho}_i(\boldsymbol{B})$, for every $i \in [1, \nu]$, each repeated d_i times.

Using the group characters

Corollary

Using the same notation as above, for each $i \in [1, \nu]$, *the eigenvalues* $\lambda_{u,j}$, *for* $u \in V$ *and* $j \in [1, d_i]$, *of the lift* Γ^α, *are the solutions* (*each repeated* d_i *times*) *of the system*

$$\sum_{u \in V, j \in [1, d_i]} \lambda_{u,j}^{\ell} = \sum_{p \in P_\ell} \chi_i(p), \qquad \ell = 1, \ldots, rd_i. \tag{1}$$

Using the group characters

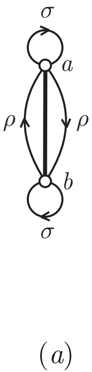
Corollary

Using the same notation as above, for each $i \in [1, \nu]$*, the eigenvalues* $\lambda_{u,j}$*, for* $u \in V$ *and* $j \in [1, d_i]$*, of the lift* Γ^α*, are the solutions (each repeated* d_i *times) of the system*

$$\sum_{u \in V, j \in [1, d_i]} \lambda_{u,j}^{\ell} = \sum_{p \in P_\ell} \chi_i(p), \qquad \ell = 1, \ldots, rd_i. \tag{1}$$

The equalities in (1) lead to a polynomial of degree rd_i, with roots the required eigenvalues $\lambda_{u,j}$.

An example



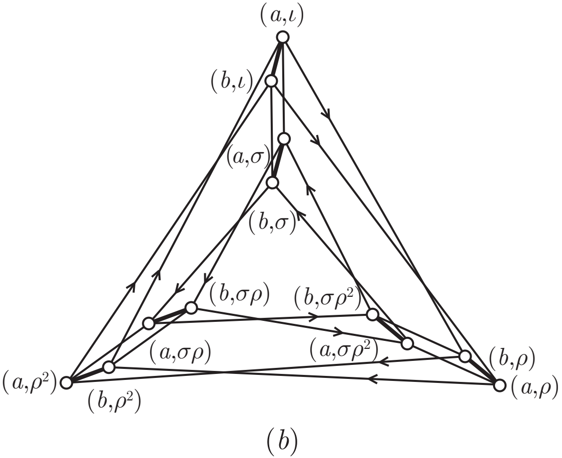


Figure: The base digraph K_2^*, on the group S_3, and its lift

$$\boldsymbol{B} = \begin{pmatrix} \sigma & \iota + \rho \\ \iota + \rho & \sigma \end{pmatrix}.$$

$S_3 \backslash g$	ι	σ	$\sigma\rho$	$\sigma\rho^2$	ρ	ρ^2
$\boldsymbol{\rho}_1$	1	1	1	1	1	1
$\boldsymbol{\rho}_2$	1	-1	-1	-1	1	1
$\boldsymbol{\rho}_3$	$\boldsymbol{I}$	$\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$	-	-	$\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$	-

Table: The irreducible representations of the symmetric group S_3.

$$\boldsymbol{B} = \begin{pmatrix} \sigma & \iota + \rho \\ \iota + \rho & \sigma \end{pmatrix}.$$

$S_3 \quad \backslash \quad g$	ι	$\sigma, \sigma\rho, \sigma\rho^2$	ρ, ρ^2
$\chi_1 \ (d_1 = 1)$	1	1	1
$\chi_2 \ (d_2 = 1)$	1	-1	1
$\chi_3 \ (d_3 = 2)$	2	0	-1

Table: The character table of the symmetric group S_3.

$$\chi_1(\boldsymbol{B}) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \chi_2(\boldsymbol{B}) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \quad \chi_3(\boldsymbol{B}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, by the Corollary:

- χ_1: Since $d_1 = 1$, two eigenvalues of Γ^α are $\{3, -1\} = \operatorname{ev} \chi_1(\boldsymbol{B})$.
- χ_2: Since $d_2 = 1$, two eigenvalues of Γ^α are $\{-3, 1\} = \operatorname{ev} \chi_2(\boldsymbol{B})$.
- χ_3: Since $d_3 = 2$, we consider all the possible closed walks of lengths $\ell = 1, 2, 3, 4$ in $\boldsymbol{B}$, which gives the system

$$\begin{aligned} \lambda_{u,0} + \lambda_{u,1} + \lambda_{v,0} + \lambda_{v,1} &= 0 \\ \lambda_{u,0}^2 + \lambda_{u,1}^2 + \lambda_{v,0}^2 + \lambda_{v,1}^2 &= 2 \\ \lambda_{u,0}^3 + \lambda_{u,1}^3 + \lambda_{v,0}^3 + \lambda_{v,1}^3 &= 0 \\ \lambda_{u,0}^4 + \lambda_{u,1}^4 + \lambda_{v,0}^4 + \lambda_{v,1}^4 &= 2, \end{aligned}$$

  with solutions $1, 0, 0, -1$

Then,

$$\operatorname{sp} \Gamma^\alpha = \{3^{(1)}, 1^{(3)}, 0^{(4)}, -1^{(3)}, -3^{(1)}\}.$$

References

L. Babai. Spectra of Cayley graphs, *J. Combin. Theory Ser. B* **27** (1979) 180–189.

E. T. Baskoro, L. Branković, M. Miller, J. Plesník, J. Ryan, J. Širáň. Large digraphs with small diameter: A voltage assigment approach. *JCMCC* **24** (1997) 161–176.

C. Dalfó, M. A. Fiol, M. Miller, J. Ryan, and J. Širáň. An algebraic approach to lifts of digraphs, submitted (2017), `arXiv:1612.08855v2 [math.CO]`.

P. J. Davis. *Circulant Matrices*. John Wiley & Sons, New York, 1979.

C. D. Godsil. *Algebraic Combinatorics*. Chapman and Hall, New York, 1993.

L. Lovász. Spectra of graphs with transitive groups, *Period. Math. Hungar.* **6** (1975) 191–196.

M. Miller, J. Širáň. Moore graphs and beyond: A survey of the degree/diameter problem. *Electron. J. Combin.* **20(2)** (2013) #DS14v2.

Thanks for your attention