Characterizing identifying codes in a digraph or graph from its spectrum

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Outlook

1. Introduction
2. Non-spectral results for digraphs
3. Spectral results for digraphs
4. The case of graphs

1. Introduction

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Definition

Given a vertex subset $U \subset V$, let $N^-[U] = \bigcup_{u \in U} N^-[u]$. For a given integer $\ell \geq 1$, a vertex subset $C \subset V$ is a $(1, \leq \ell)$-identifying code in a digraph D when, for all distinct subsets $X, Y \subset V$, with $1 \leq |X|, |Y| \leq \ell$, we get

$$N^-[X] \cap C \neq N^-[Y] \cap C.$$

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$$N^-[X] \cap C \neq N^-[Y] \cap C.$$

Remark

A digraph $D = (V, A)$ admits some $(1, \leq \ell)$-identifying code if and only if for all distinct subsets $X, Y \subset V$ with $|X|, |Y| \leq \ell$, we have

$$N^-[X] \neq N^-[Y].$$

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Two distinct vertices u and v of D are called *twins* if $N^-[u] = N^-[v]$.

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Remark

A digraph D admits a $(1, \leq 1)$-identifying code if and only if D is twin-free.

2. Non-spectral results for digraphs

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Theorem

Every 1-in-regular digraph D admits a $(1, \leq 2)$-identifying code if and only if the girth of D is at least 5.

2. Non-spectral results for digraphs

Theorem (Balbuena, D., Martínez-Barona, 2017)

Let D be a 2-in-regular digraph. Then,

(i) *D admits a $(1, \leq 1)$-identifying code if and only if it does not contain any subdigraph isomorphic to the digraph of Figure 1 (a).*

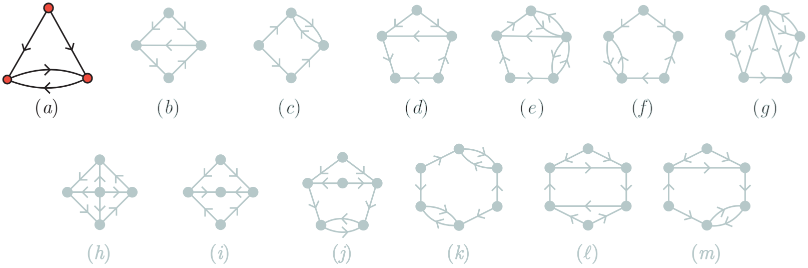


Figure 1

2. Non-spectral results for digraphs

Theorem (Balbuena, D., Martínez-Barona, 2017)

Let D be a 2-in-regular digraph. Then,

(ii) *D admits a $(1, \leq 2)$-identifying code if and only if it does not contain any subdigraph isomorphic to any of the digraphs of Figure 1.*

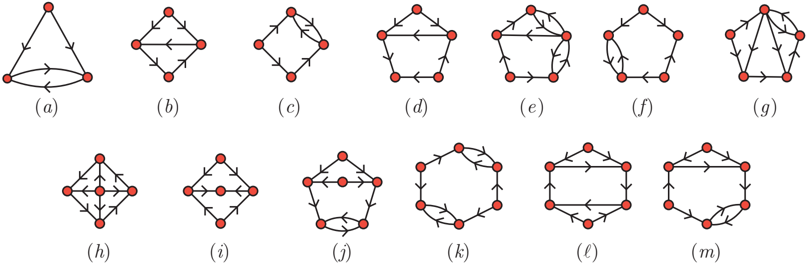


Figure 1

2. Non-spectral results for digraphs

Corollary

Every oriented and TT_3-free 2-in-regular digraph admits a $(1, \leq 2)$-identifying code if and only if it does not contain any subdigraph isomorphic to Figure 1 (i).

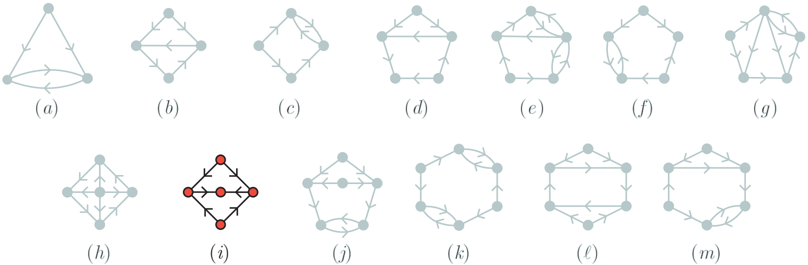
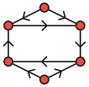


Figure 1

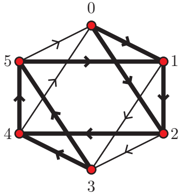
2. Non-spectral results for digraphs

Example

Circulant digraph $C(6;1,2)$.
$X = \{0,3\}$ and $Y = \{1,4\}$, $N^{-}[X] = N^{-}[Y] = \{0,1,2,3,4,5\} = V$,
$C(6;1,2)$ cannot admit a $(1, \leq 2)$-identifying code.
Subdigraph of Figure 1 $(\ell) \subset C(6;1,2)$.



1 (ℓ)

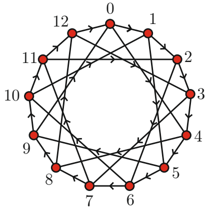


$C(6;1,2)$

2. Non-spectral results for digraphs

Example

Circulant digraph $C(13; 1, 4)$.
$C(13; 1, 4)$ is oriented, TT_3-free.
By the Corollary, $C(13; 1, 4)$ admits a $(1, \leq 2)$-identifying code. Since $N^-[X] \neq N^-[Y]$ for every two distinct sets X, Y such that $2 \leq |X|, |Y| \leq 3$, $C(13; 1, 4)$ admits a $(1, \leq 3)$-identifying code.



2. Non-spectral results for digraphs

Theorem (Balbuena, D., Martínez-Barona, 2017)

Let D be a 2-in-regular digraph. D admits a $(1, \leq 3)$-identifying code if and only if it is oriented, TT_3-free and does not contain any subdigraph isomorphic to any of the digraphs of Figure 2.

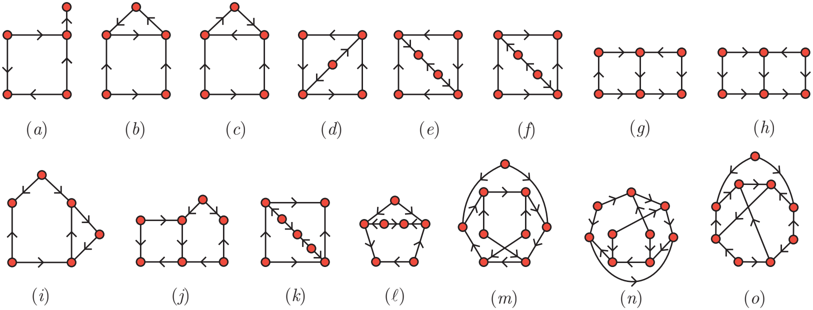


Figure 2

2. Non-spectral results for digraphs

Lemma

Let D be a d-in-regular digraph on n vertices, without any of the subdigraphs of Figure 3. If D admits a $(1, \leq \ell)$-identifying code, then $\ell \in \{d, d+1\}$.

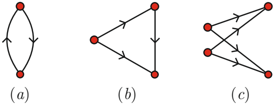


Figure 3

3. Spectral results for digraphs

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Lemma

Let D be a digraph with adjacency matrix A and with a set of eigenvalues denoted by $\mathrm{ev}(A)$. If $-1 \notin \mathrm{ev}(A)$, then D admits a $(1, \leq 1)$-identifying code.

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Let D be a digraph with adjacency matrix A and with a set of eigenvalues denoted by $\mathrm{ev}(A)$. If $-1 \notin \mathrm{ev}(A)$, then D admits a $(1, \leq 1)$-identifying code.

Remark

The converse is not true.

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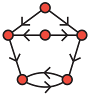
Remark

The converse is not true.

Example

The digraph of the figure has -1 as an eigenvalue, but it does admit a $(1, \leq 1)$-identifying code.



3. Spectral results for digraphs

Lemma

Let D' be a digraph with maximum in-degree Δ^- having an eigenvalue λ with eigenvector $\boldsymbol{x}' = (x'_u)$, such that $x'_v = 0$ for any vertex $v \in V(D')$ with $d^-(v) < \Delta^-$. Then, any Δ^--in-regular digraph D containing D' as a subdigraph has also the eigenvalue λ.

3. Spectral results for digraphs

Theorem

Let D be a 2-in-regular *digraph with adjacency matrix A.*

(i) *If $-1 \notin \mathrm{ev}(A)$ and D does not contain any subdigraph isomorphic to (b), (c), (d), (f) and (i) of Figure 1, then D admits a $(1, \leq 2)$-identifying code.*

$$-1 \notin \mathrm{ev}(A)$$

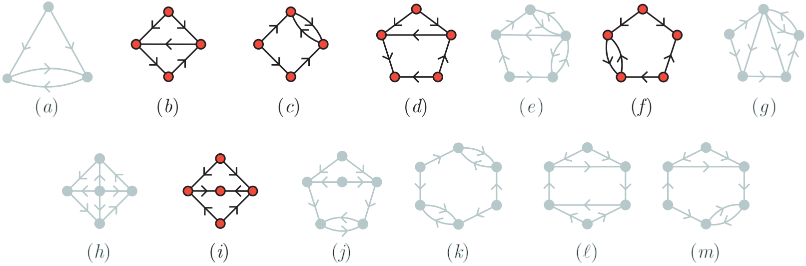


Figure 1

3. Spectral results for digraphs

Theorem

Let D be a 2-in-regular digraph with adjacency matrix A.

(ii) *If $-1, 0 \notin \mathrm{ev}(A)$ and D does not contain any subdigraph isomorphic to (b)-(ℓ) of Figure 2, then D admits a $(1, \leq 3)$-identifying code.*

$$-1{,}0 \notin \mathrm{ev}(A)$$

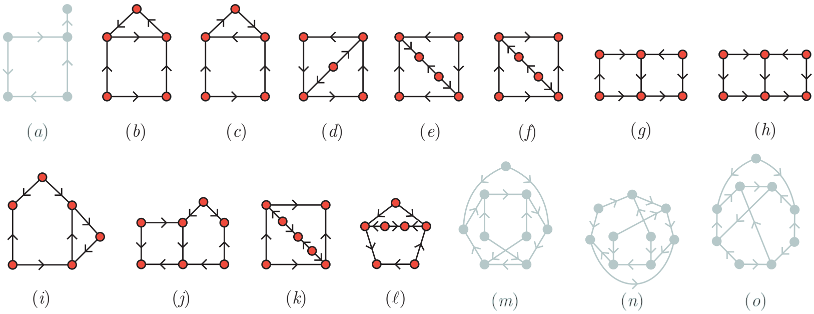


Figure 2

3. Spectral results for digraphs

Proposition

Let $D = (V, E)$ be a digraph with adjacency matrix A having some real eigenvalue, say $\lambda \in \mathrm{ev}(A)$, different from the spectral radius. Let $\boldsymbol{x} = (x_u)_{u \in V}$ be an eigenvector of A associated to λ such that $X = \mathcal{P}(\boldsymbol{x}) = \{i : x_i > 0\}$ and $Y = \mathcal{N}(\boldsymbol{x}) = \{i : x_i < 0\}$.

(a) If $\lambda < 0$, then $X \cup N^-(X) = Y \cup N^-(Y) \quad (\Leftrightarrow N^-[X] = N^-[Y])$.

(b) If $\lambda > 0$, then $X \cup N^-(Y) = Y \cup N^-(X)$.

(c) If $\lambda = 0$, then $N^-(X) = N^-(Y)$.

3. Spectral results for digraphs

Corollary

Let D be a digraph admitting a $(1, \leq \ell)$-identifying code. Let A be its adjacency matrix having at least one negative eigenvalue $-\lambda$ (for $\lambda > 0$) with $\boldsymbol{x} = (x_1, \ldots, x_n)$ any associated eigenvector. Then

$$\ell < \min_{\boldsymbol{x}} \max\{|\mathcal{P}(\boldsymbol{x})|, |\mathcal{N}(\boldsymbol{x})|\}.$$

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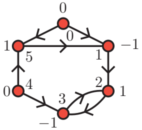
Example

$\mathrm{sp}(A) = \{0^4, 1^1, -1^1\}$.
Eigenvector corresponding to -1: $(0, -1, 1, -1, 0, 1)$.
$X = \{2, 5\}$, and $Y = \{1, 3\}$.
$N^-[X] = N^-[Y] = \{0, 1, 2, 3, 4, 5\}$.
It does not admit a $(1, \leq 2)$-identifying code.



3. Spectral results for digraphs

Lemma

Let D be a digraph on n vertices with adjacency matrix A, and let λ be a real eigenvalue of A with geometric multiplicity m. For any given index set $I \subset \{1, 2, \ldots, n\}$ with $|I| = m - 1$, there exists an eigenvector $\boldsymbol{x}$ with eigenvalue λ and entries $x_i = 0$ for every $i \in I$.

4. The case of graphs

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Remark

The last Corollary can also be applied to graphs, which always have real eigenvalues.

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Theorem (Laihonen, 2008)

Let $k \geq 2$ be an integer.

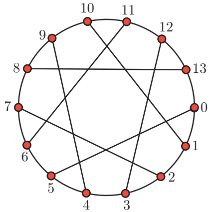
1. *If a k-regular graph has girth $g \geq 7$, then it admits a $(1, \leq k)$-identifying code.*
2. *If a k-regular graph has girth $g \geq 5$, then it admits a $(1, \leq k-1)$-identifying code.*

Example

Heawood graph: $\mathrm{sp}(H) = \{3^1, \sqrt{2}^6, -\sqrt{2}^6, -3^1\}$.
Eigenvectors:

$$\begin{aligned}
\lambda = -3: &\quad (-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1);\\
\lambda = -\sqrt{2}: &\quad (-1,0,1,0,-1,\sqrt{2},0,-\sqrt{2},1,0,0,0,0,0),\\
&\quad (-1,0,1,-\sqrt{2},0,\sqrt{2},-1,0,0,0,0,0,1,0),\\
&\quad (0,-1,\sqrt{2},-1,0,1,-\sqrt{2},0,0,0,0,1,0,0),\\
&\quad (0,-1,\sqrt{2},0,-\sqrt{2},1,0,-1,0,1,0,0,0,0),\\
&\quad (0,-\sqrt{2},1,0,-1,\sqrt{2},-1,0,0,0,1,0,0,0),\\
&\quad (-\sqrt{2},0,\sqrt{2},-1,0,1,0,-1,0,0,0,0,0,1)
\end{aligned}$$

With the last one: $X = \{2,5,13\}$, $Y = \{0,3,7\}$.
$N^-[X] = N^-[Y] = \{0,1,2,3,4,5,6,7,8,12,13\}$.
It does not admit a $(1, \leq 3)$-identifying code.
Laihonen's Theorem: It is 3-regular and it has girth 6, it admits a $(1, \leq 2)$-identifying code.



Thank you for your attention.

Gràcies per la vostra atenció.