Average mixing matrix

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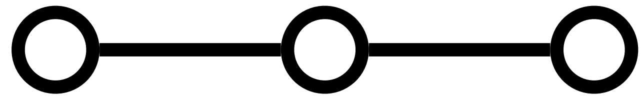
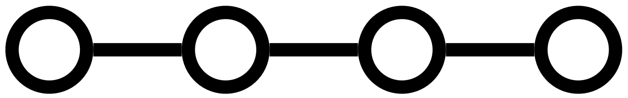
The matrix

Adjacency matrix $$A = \sum_{r=0}^{d} \theta_r E_r$$

Average mixing matrix $$\widehat{M} = \sum_{r=0}^{d} E_r \circ E_r$$

Examples





$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \begin{pmatrix} \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix} \qquad \begin{pmatrix} \frac{3}{10} & \frac{1}{5} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{3}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{5} & \frac{3}{10} \end{pmatrix}$$

Examples

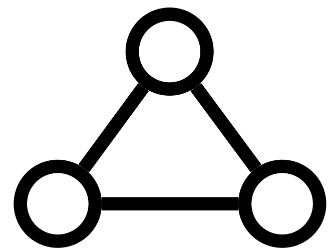
Theorem (Godsil 2013)

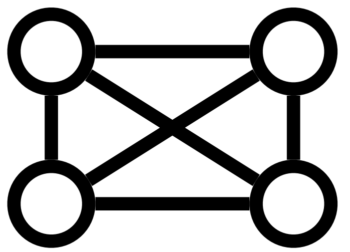
The average mixing matrix for the path on n vertices is

$$\widehat{M} = \frac{1}{2n+2}(2J + I + R)$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix} \quad \begin{pmatrix} \frac{3}{10} & \frac{1}{5} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{3}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{5} & \frac{3}{10} \end{pmatrix}$$

Examples





$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{5}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{5}{9} \end{pmatrix} \qquad \begin{pmatrix} \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{5}{8} \end{pmatrix}$$

Examples

Theorem (Al et Tamon 2003)

For the complete graph, we have

$$\widehat{M} = \frac{1}{n^2}(2J + [(n-1)^2 + 1]I)$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{5}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{5}{9} \end{pmatrix} \quad \begin{pmatrix} \frac{5}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{5}{8} \end{pmatrix}$$

Properties

$$A = \sum_{r=0}^{d} \theta_r E_r \qquad \widehat{M} = \sum_{r=0}^{d} E_r \circ E_r$$

The average mixing matrix is.....

Non-negative

Positive iff graph is connected

Rational (Godsil 2013)

Constant iff the graph is K_2

Motivation

Continuous-time quantum walk $\exp(\mathrm{i}tA)$

Mixing matrix (probabilities) $\exp(\mathrm{i}tA) \circ \overline{\exp(\mathrm{i}tA)}$

The average

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \exp(\mathrm{i}tA) \circ \overline{\exp(\mathrm{i}tA)} \, \mathrm{d}t$$

$$= \sum_{r=0}^{d} E_r \circ E_r$$

More properties

Each row is an average of a probability distribution......

Doubly-stochastic

It is a sum of Schur squares of PSD matrices....

Positive semidefinite

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What else?

What does it say about the graph?

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Completely positive semidefinite

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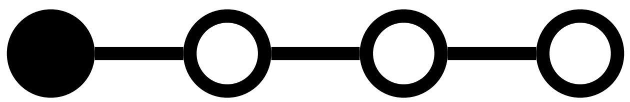
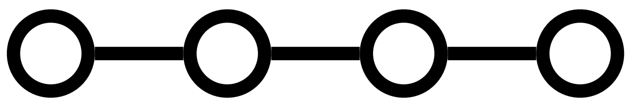
Completely positive semidefinite



$$D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\phi(D_1) = \sum_{r=0}^{d} E_r D_1 E_r = \begin{pmatrix} \frac{3}{10} & 0 & -\frac{1}{10} & 0 \\ 0 & \frac{1}{5} & 0 & -\frac{1}{10} \\ -\frac{1}{10} & 0 & \frac{1}{5} & 0 \\ 0 & -\frac{1}{10} & 0 & \frac{3}{10} \end{pmatrix}$$

Completely positive semidefinite



$$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

↓ projection onto commutant of A ↓

$$\phi(D_1) \quad \phi(D_2) \quad \phi(D_3) \quad \phi(D_4)$$

THEOREM (COUTINHO, GODSIL, GUO, ZHAN' 17+)

The average mixing matrix is the Gram matrix of the matrices $\phi(D_i)$

Consequence

Let $\mathcal{D}$ be the space of all diagonal matrices.

Let $\phi(\mathcal{D})$ be the projection of this space onto the commutant of the adjacency matrix of the graph.

$$\texttt{rk}(\widehat{M}) = \texttt{dim}\ \phi(\mathcal{D})$$

We can decompose the commutant

$$\texttt{Comm}(A) = \phi(\mathcal{D}) \oplus \phi(\mathcal{D})^{\perp}$$

Permutation matrices are examples of non-obvious matrices that live in the commutant of the adjacency matrix.

Consequence

Consider the vertices for which there is an automorphism fixing only that vertex...

Theorem (Coutinho, Godsil, Guo, Zhan' 17+)

The rank of $\widehat{M}$ upper bounds the number of such vertices.

Theorem (Coutinho, Godsil, Guo, Zhan' 17+)

If $A(G)$ has simple eigenvalues, then $\mathtt{rk}(\widehat{M}) \leq n - 1$.

If equality, then any automorphism of G has fixed points.

Bottom line

(1) Quantum walk at continuous time

(2) Probability distribution at each instant

(3) Average of these probabilities

(4) Matrix defined only in terms of the eigenspaces

(5) Satisfying many interesting properties

(6) Closely related to the commutant algebra of the graph

(7) Consequences to the symmetries of the graph

(8) What else?

Trace? Eigenvalues? Eigenvectors? Entropy?

Completely positive semidefinite rank?

The end

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