Parameter Estimation for Mixed-Mechanism Tear Film Thinning

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Introduction
Tear film breakup (TBU) occurs when a dry spot forms on the eye. Evaporation causes relatively slow thinning [1]; lipid-driven tangential flow is hypothesized to cause rapid breakup [2]. We implement a model for a mixture of these mechanisms and fit to breakup occurring on an intermediate time scale and contrast with purely evaporative results. Ranges for parameters affecting tear film (TF) thickness and fluorescent (FL) intensity distributions over time are not well known, our estimates are comparable to experimental values.

Goals
- Determine experimental quantities that cannot be measured in vivo during TBU by parameter identification via fitting to our models.
- Compare our mixed-mechanism and evaporative thinning models [3].

Cartesian Model
We model TF thickness, \( h(x,t) \), pressure, \( p(x,t) \), osmolarity, \( c(x,t) \), FL concentration, \( f(x,t) \), and surface concentration of lipid \( \Gamma(x,t) \), and calculate FL intensity \( I(h,f) \):

\[
\partial_t h = -\partial_x (h u) + P_t (c - 1) - J, \quad 0 < x < X_0
\]

\[
\partial_t \Gamma = [P_e c \partial_x \Gamma - \partial_t (u \Gamma)] B
\]

\[h(\partial_c + \bar{u} \partial_x c) = P_e \bar{u} h \partial_x c - P_t (c - 1) c + J c\]

\[h(\partial_f + \bar{u} \partial_x f) = P_e \bar{u} h \partial_x f - P_t (c - 1) f + J f\]

\[p = -\partial_x h, \quad I = h |1 - e^{-\phi t}|^{1 + f^2}
\]

where \( J \) is evaporation, \( \bar{u} \) is the depth-averaged fluid velocity, \( u_s \) is the surface velocity, \( B \) is a smooth transition function, and \( \phi, P_t, P_e, \bar{u}, c, f \) are constants. No flux at \( x = 0, x = X_0 \).

Optimization

\[
\arg \min_{c(x,t)} \| I_{th}(x,t) - I_{ex}(x,t) \|^2
\]

where the parameters are
- \( v \), rate of evaporation (in \( \mu m/min \)),
- \( X_f \), width of glob (in mm),
- \( (\Delta \sigma)_h \), change in surface tension (in \( \mu N/m \)), with experimental FL intensity \( I_{ex} \) (Fig. 1) and \( I_{th} \) computed from our Cartesian model.

We use three choices for the evaporation \( J \): zero, uniform, and high under the glob, zero outside.

Methods
- The tear film is modeled as a single-layer incompressible Newtonian fluid.
- We derive a system of equations for TF thickness, pressure, osmolarity, FL concentration and surfactant concentration via lubrication theory.
- The system is nondimensionalized using an evaporative time scale and discretized in space using Chebyshev spectral collocation, resulting in DAEs (differential algebraic equations) solved via Matlab’s ode15s.
- Extracted FL intensity data from every time level across a spot or streak TBU is fit with the model (see Fig. 1) using Matlab’s lsqnonlin (Levenberg-Marquardt algorithm).

Nondimensional Model Solutions

Figure 2: A surface tension based time scale is used. Tangential flow drives extreme thinning in the streak center. Similarities between \( f \) and \( h \) allow inferences about \( h \) from exp. intensity data. Arrows show increasing time (final time: 0.2 s).

Mixed-Mechanism Fit Results

Evaporative Fit Results

Comparison of Fig. 3 and Fig. 5

Conclusions & Next Steps
- Intermediate fits are improved with nonzero evaporation, evidence that tangential flow and evaporation cooperate to cause breakup.
- Most instances of TBU are mixed-mechanism and the model helps decipher how much of each is present.
- We are fitting more cases of rapid lipid-driven thinning and will move to 2D models.

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References