Introduction
A graph signal is a function \( f: V \to \mathbb{C} \), where \( V \) is the vertex set of a graph. With increasing amounts of data being recorded which naturally embeds in a graph structure, there is growing interest in generalizing tools of classical signal processing to this setting. See [1] for a discussion of generalizing the DFT to this setting. For an introduction more focused on generalizing traditional signal processing applications, see [2].

Today I discuss recent work on the problem of characterizing graph signals with well-localized translations which arose in generalizing the short-term Fourier transform. In particular, I explain the use of representation theory to describe these functions for many families of Cayley graphs.

Graph Signal Fourier Transform
The graph Laplacian is \( L = D - A \), with \( D \) the diagonal degree matrix and \( A \) the adjacency matrix. If we fix a basis of eigenvectors \( \Phi \) of \( L \), then the Fourier transform of a graph signal \( f \) is the expansion of \( f \) in terms of \( \Phi \). That is, for a graph of order \( N \),

\[
\hat{f}(\lambda) = \langle f, \phi_{\lambda} \rangle = \sum_{i=1}^{N} f(v_i)\phi_{\lambda}(v_i).
\]

In this setting, the inverse Fourier transform is given by

\[
f(v) = \sum_{\lambda \in \Lambda} \hat{f}(\lambda)\phi_{\lambda}(v_i) = \frac{1}{N} \sum_{\lambda \in \Lambda} \hat{f}(\lambda)\phi_{\lambda}(v_i).
\]

If we think of \( f \) and \( \hat{f} \) as column vectors and \( \Phi \) as the matrix of basis vectors, then these definitions naturally lend themselves to the notation

\[
\hat{f} = \Phi^T f, \quad \text{and} \quad f = \Phi \hat{f}.
\]

Graph Signal Translation
The graph translation operator is defined by convolution with the Kronecker delta function \( \delta_i \) and then by taking the inverse Fourier transform:

\[
(T_f(v)) = \sqrt{N} \delta_i\hat{f}(v) = \sqrt{N} \sum_{\lambda \in \Lambda} \hat{f}(\lambda)\phi_{\lambda}(v_i)\phi_{\lambda}(v_k).
\]

Remark: One of the chief difficulties in the graph setting is the lack of regularity in both the vertex and spectral domains. For example, in particular, I explain the use of representation theory to describe these functions for many families of Cayley graphs.

Window Functions and Previous Work
Let \( G \) be a graph of order \( N \). A window function is \( f: V \to \mathbb{C} \) such that \( T_f(v_i) = 0 \) when \( d(v_i, v_k) > r \) for some integer \( 0 < r < N \). Here we use the geodesic distance as our metric.

Previous Work: The authors prove in [3] that if \( f \) is a polynomial of degree \( r \), then \( f \) will be a window function. This shows that the dimension of the space of window functions is at least \( r + 1 \).

Our Contribution: Using representation theory, we fully classify window functions on Cayley graphs generated by the union of conjugacy classes. To do this, we exploit the fact that these graphs have an eigenbasis formed by the coordinate functionals of the group’s irreducible representations. These details are provided in the next column.

New Theorem
Let \( G = \langle S \rangle \) be a finite group. Let \( G = \text{cay}(G, S) \) be the Cayley graph generated by \( S \subseteq G \) where \( S = \bigcup_{i \in I} C_i \) and each \( C_i \) is a conjugacy class. Let \( \eta \) denote the identity element, and let \( \{h_i\}_{i=1}^m \) be a complete set of representatives for the conjugacy classes of \( G \). Let \( \chi \) be the standard character table of \( G \) with columns \( \phi_i \), corresponding to the conjugacy class containing \( h_i \). Also denote the characters by \( \{\chi_i\}_{i=1}^m \). Then \( f \) is a window function with \( T_f(v_i) \) sharply localized in the ball of distance \( k \) centered at vertex \( v_i \) if and only if \( \hat{f}(\phi_i) = 0 \).

Further, given the subset \( S_i = \{i \in I | d(v_i, \eta) \leq k\} \), we can construct an orthogonal basis for the space of window functions on \( G \) localized in the \( k \)-ball by lifting the vector \( (\chi_i)_{i=1}^m \) from \( \mathbb{C}^m \) to \( \mathbb{C}^N \) and then taking their Fourier inverse. That is, \( T_{\hat{f}}(v_i) = 0 \) for all \( d(v_i, v_k) > k \) if and only if \( \hat{f} \) satisfies the span condition given to the right. Note that \( \chi(e) \) is the degree of the representation \( \chi \), and each block is size \( \chi(e) \) by \( 1 \).

Main Idea of Proof: Given the basis of coordinate functionals, we can rewrite the translation operator as

\[
(T_f(v_i)) = \frac{1}{\sqrt{N}} \sum_{\lambda \in \Lambda} \chi_i(\lambda)f(\lambda)e^{-i\theta k},
\]

which reduces the problem to finding orthogonality relationships in the character table of the underlying group.

An Example: The Cayley Graph of \( S_4 \) Generated by Transpositions
The graph \( G = \text{cay}(S_4, \{(12), (13), (14), (23), (24), (34)\}) \) is pictured to the left. The character table of \( S_4 \) is given in the next column. Let’s say that we want to find a basis for all functions \( f: V \to \mathbb{C} \) on this graph such that \( T_f(v_i) = 0 \) if \( d(v_i, v_k) > 2 \). Then \( f \) must be orthogonal to the last column of the character table in \( C_4 \), but we can lift this to \( C_8 \) using the formula above to determine that

\[
\hat{f} \in \text{span}_\mathbb{C} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.
\]

Here we use the superscript notation \( (n) \) to denote that the size of the blocks are \( n \) by \( 1 \).

Eigenvectors of the Graph Laplacian
Let \( G = \text{cay}(G, S) \) with adjacency matrix \( A \). Then for \( \rho \in \mathbb{C} \),

\[
A\rho = \lambda \rho,
\]

where \( \lambda_r \) is the coordinate functional for the representation \( \rho \) and \( \lambda_i = \frac{1}{\sqrt{N}} \sum_{v \in V} x_i(v) \). We are able to apply this result to the graph Laplacian as it is a polynomial of the adjacency matrix.

Character Table of \( S_4 \)

<table>
<thead>
<tr>
<th>Character</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
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<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

Remark: Using the character table instead of the coordinate functionals when computing the space of window functions yields a substantial decrease in complexity.

Comparison of Results
The theorem in [3] shows that for \( S_i \) generated by conjugacy classes, \( f \) is sharply localized if \( \hat{f} \) is a polynomial, or equivalently,

\[
\hat{f} \in \text{span}_\mathbb{C} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.
\]

It is easy to verify that this forms a 3-dimensional subspace of the 4-dimensional space of window functions for this graph found using representation theory. It is also worth noting that the proof technique used in [3] provides no means of generalization to finding other sharply localized functions.

Future Work
Continuing work on this project will include
- classifying window functions for Cayley graphs with arbitrary generating sets.
- studying the relationship between the Discrete Fourier Transform and the Graph Fourier Transform. (In the case of finite abelian groups, they are identical)
- generalizing the results from finite groups to locally compact groups.

References

http://math.udel.edu/