Non-Destructive Testing of Anisotropic Materials and Transmission Eigenvalues

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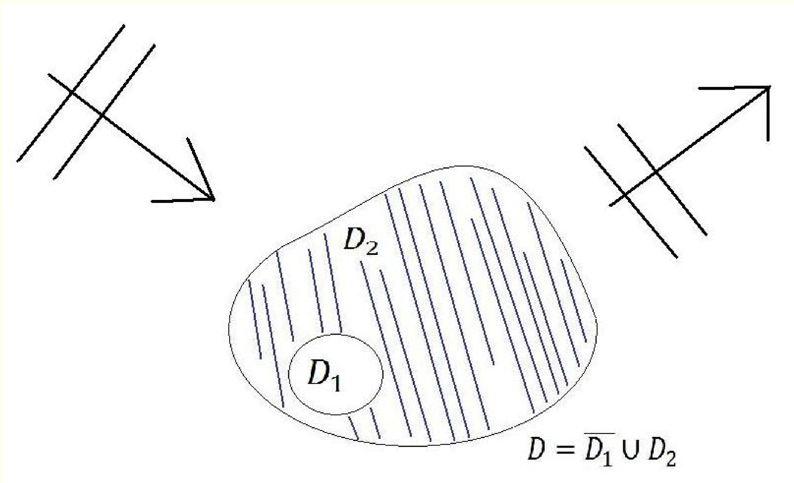
Introduction

I am working on the inverse scattering problem for anisotropic inhomogeneous media with voids. My main goal is to study the inverse problem of determining the material properties of the anisotropic medium and/or the size and location of the void. This is an important Applied Mathematics problem since it arises in nondestructive testing of exotic materials.

Direct Problem

Let $D \subset \mathbb{R}^2$ be a bounded simply connected open set with a Lipshitz boundary and let $\overline{D}_1 \subset D$ with a boundary that is also Lipshitz.



- Let $A(x)$ be a symmetric matrix with entries $a_{ij}(x) \in L^\infty(D \setminus \overline{D}_1)$
- $\overline{\xi} \cdot \Re(A)\xi \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and $x \in D \setminus \overline{D}_1$
- $\overline{\xi} \cdot \Im(A)\xi \leq 0$ for all $\xi \in \mathbb{C}^2$ and $x \in D \setminus \overline{D}_1$
- Assume $n \in L^\infty(D \setminus \overline{D}_1)$ such that $\Im(n) \geq 0$.

The Formulation of the Direct Problem is:

Find $(u^s, w) \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D}) \times H^1(D)$ such that:

$$\Delta w + k^2 w = 0 \quad \text{in} \quad D_1 \qquad (1)$$
$$\nabla \cdot A(x)\nabla w + k^2 n(x) w = 0 \quad \text{in} \quad D_2 \qquad (2)$$
$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D} \qquad (3)$$
$$w^- = w^+ \text{ and } \frac{\partial w^-}{\partial \nu} = \frac{\partial w^+}{\partial \nu_A} \quad \text{on} \quad \partial D_1 \qquad (4)$$
$$w - u^s = e^{ikx\cdot d} \quad \text{on} \quad \partial D \qquad (5)$$
$$\frac{\partial w}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial e^{ikx\cdot d}}{\partial \nu} \quad \text{on} \quad \partial D \qquad (6)$$
$$\lim_{r\mapsto\infty} \sqrt{r}\Big(\frac{\partial u^s}{\partial r} - iku^s\Big) = 0 \qquad (7)$$

Theorem 1: *There exists a unique solution to the BVP (1)-(7) depending continuously on the data.*

Far-Field Operator

The scattering field u^s has the asymptotic expansion:

$$u^s(r,\theta) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\theta,\phi) + \mathcal{O}(r^{-3/2}) \quad \text{as} \quad r \to \infty$$

u_∞ is called the far field pattern of the scattering problem. With ϕ incident direction and θ observation direction. We now define the far-field operator:

$$(Fg)(\theta) := \int_0^{2\pi} u_\infty(\theta,\phi) g(\phi)\, d\phi$$

The associated Herglotz function is of the form:

$$v_g(x) := \int_0^{2\pi} e^{ikx\cdot d} g(\phi)\, d\phi. \text{ were } d = (\cos\phi, \sin\phi)$$

The Inverse Problem is to infer information about the domain and/or coefficients of the BVP (1)-(7) given the far field operator.

Theorem 2: *The far field operator corresponding to the scattering problem (1)-(7) is injective with dense range iff $\nexists$ nontrivial (w, v_g) solving:*

$$\Delta w + k^2 w = 0 \quad \text{in} \quad D_1$$
$$\nabla \cdot A\nabla w + k^2 n w = 0 \quad \text{in} \quad D_2$$
$$\Delta v_g + k^2 v_g = 0 \quad \text{in} \quad D$$
$$w^- = w^+ \text{ and } \frac{\partial w^-}{\partial \nu} = \frac{\partial w^+}{\partial \nu_A} \quad \text{on} \quad \partial D_1$$
$$w = v_g \text{ and } \frac{\partial w}{\partial \nu_A} = \frac{\partial v_g}{\partial \nu} \quad \text{on} \quad \partial D$$

Transmission Eigenvalues (TEVs)

The interior TEVs are $k \in \mathbb{C}$ s.t. $\exists$ nontrivial (w, v):

$$\Delta w + k^2 w = 0 \quad \text{in} \quad D_1$$
$$\nabla \cdot A\nabla w + k^2 n w = 0 \quad \text{in} \quad D_2$$
$$\Delta v + k^2 v = 0 \quad \text{in} \quad D$$
$$w^- = w^+ \text{ and } \frac{\partial w^-}{\partial \nu} = \frac{\partial w^+}{\partial \nu_A} \quad \text{on} \quad \partial D_1$$
$$w = v \text{ and } \frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial D$$

We are interested in the TEVs since the eigenvalues hold information about A, n and the void D_1.

Theorem 3: *Assume that $A - I$ and $n - 1$ have different signs then the set of TEVs is non-empty, provided $|D_1|$ is sufficiently small.*

Theorem 4: *Assume that $D_1 \subseteq D_1'$ with D, A and n fixed. Also let $A - I$ and $n - 1$ have different signs. Then we have that the first TEV is an increasing function of void size:*

$$k_1(D_1) \leq k_1(D_1')$$

TEVs in the Data

The radiating fundamental solution to Helmholtz equation in $\mathbb{R}^2$ is given by $\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|)$, and let $\Phi_\infty(\cdot,\cdot)$ be the far field pattern of $\Phi(x,y)$.

We assume that in some neighborhood $\mathcal{N}_\delta(\partial D)$ of the boundary ∂D that either:

- $A(x) \leq A^\star I < I$ and $n(x) \leq n^\star < 1$
- $A(x) \geq A_\star I > I$ and $n(x) \geq n_\star > 1$

Then we have the following result.

Theorem 5: *Let k be a Real Transmission Eigenvalue and assume that:*

$$\lim_{\delta\to 0} ||Fg_{z,\delta} - \Phi_\infty(\cdot,z)||_{L^2(0,2\pi)} = 0.$$

Then for almost every $z \in D$, $\lim_{\delta\to 0} ||g_{z,\delta}|| = \infty$.

This result suggests that if we solve the far field equation and plot $||g||_{L^2(0,2\pi)}$ verses $[k_{min}, k_{max}]$ then we should see spikes at the TEVs.

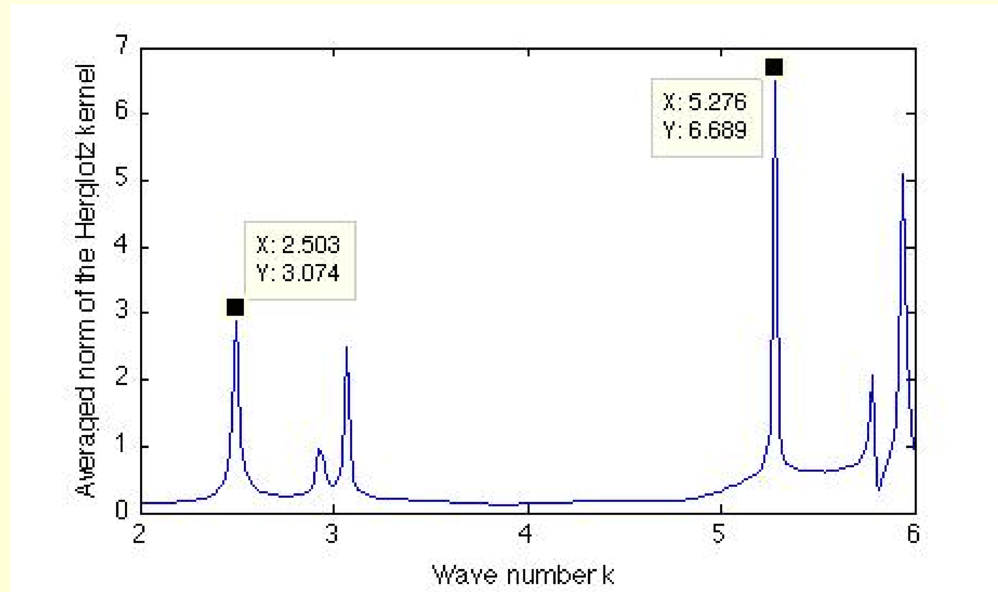
Numerical Experiments

Let $0 < \epsilon < 1$ with $D = B(0;1)$ and $D_1 = B(0;\epsilon)$.

We see the monotonicity of the 1st TEV w.r.t. the size of the void in the chart below for $n = 1$:

	$\epsilon = 0.2$	$\epsilon = 0.1$	$\epsilon = 0.05$	$\epsilon = 0.01$
$A = \frac{1}{5}I$	2.534	2.4887	2.4852	2.4849
$A = \frac{1}{2}I$	8.7883	8.0852	7.9924	7.9844

Using separation of variables one can see that $k = 2.4887, 5.2669$ are TEVs for $A = \frac{1}{5}I$ and $\epsilon = 0.1$. Below is a plot of $||g||_{L^2(0,2\pi)}$ over the interval $[2,6]$.



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