Uniqueness for an Inverse Problem with Formally Determined Offset Data
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Motivation
The problem under consideration arises in the study of geophysics and medical imaging. Let \( x = (x_1, x_2, z) \) be a point in \( \mathbb{R}^3 \), \( a = (a_1, a_2, 0) \) and \( h = (h_0, h_0, 0) \) for some \( h > 0 \). We consider an acoustic medium occupying the half space \( z \leq 0 \) and let \( q(x) \) represent some acoustic property of the medium (e.g., oil, minerals, a tumor).

The medium is probed by an acoustic wave, generated at \( a - h \), and the medium response, \( u^a \), is measured at the offset boundary location \( a + h \), for every \( a \) on the boundary \( z = 0 \). The goal is to recover the acoustic property \( q(x) \) given \( u^a(a + h, t) \) for all \( a \) on \( z = 0 \) and for all time \( t \).

Progressing Wave Expansion
Given a real valued \( q = q(x) \) on \( z \leq 0 \) representing the acoustic property of the medium, emit a acoustic wave at \( x = a - h \), characterized by \( u^a(x, t) \). It is known that \( U^a \) satisfies the following PDE:

\[
U^a_x - \Delta U^a - qU^a = 0, \quad x \in \mathbb{R}^3, \quad z \leq 0, \quad t \in \mathbb{R} \tag{1}
\]

\[
\partial_t U^a(x, t) = \delta(x - a + h, t), \quad \{z = 0\}, \quad t \in \mathbb{R} \tag{2}
\]

\[
U^a(x, t) = 0, \quad x \in \mathbb{R}^3, \quad z \leq 0, \quad t < 0 \tag{3}
\]

We extend \( U^a \) to an even function in \( z \), called \( V^a \). Then

\[
V^a_x - \Delta V^a - qV^a = \delta(x - a + h, t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R} \tag{4}
\]

\[
V^a(x, t) = 0, \quad x \in \mathbb{R}^3, \quad t < 0 \tag{5}
\]

Using the "progressing wave expansion" technique, we get

\[
V^a(x, t) = \frac{1}{4\pi} \int_{|x - a + h|} \delta(t - (x - a + h)) + v^a(x, t),
\]

where \( v^a(x, t) = 0 \) outside of the cone \( t = |x - a + h| \) and inside it is the solution of the Goursat problem:

\[
v^a_x - \Delta v^a - qv^a = 0, \quad x \in \mathbb{R}^3, \quad t \geq |x - a + h| \tag{6}
\]

\[
v^a(x, |x - a + h|) = \frac{1}{8\pi} \int_0^t q(a - h + s(x - a + h))ds, \quad x \in \mathbb{R}^3 \tag{7}
\]

Inverse Problem
Inverse Problem: Given measured data \( v^a(a + h, t) \), is the coefficient \( q(x) \) unique? I.e., Given two solutions of (6)-(7) \( v^a_1(a + h, t) \) and \( v^a_2(a + h, t) \) that yield the same measured data \( v^a_1(a + h, t) = v^a_2(a + h, t) \), are the corresponding \( q_1 \) and \( q_2 \) equal as well?

To investigate this, we require the condition in the following theorem:

**Theorem 1.** If \( v^a_1(a + h, t) = v^a_2(a + h, t) \) for all \( a \in \{z = 0\} \) and \( t \in \mathbb{R} \), then \( q_1 = q_2 \) provided there is a constant \( C \), independent of \( z \) such that

\[
||\nabla_x (q_1 - q_2)(\cdot, z)||_{L^2(\mathbb{R}^2)} \leq C||p(x_1 - q_2)(\cdot, z)||_{L^2(\mathbb{R}^2)}, \quad \forall z \in (0, 1), \tag{8}
\]

where \( \nabla_x = e_1(h + e_2) \), the gradient in the first two coordinates.

Method
**Physical Data:** \( v^a(a + h, t) \) for \( 0 \leq t \leq 2\tau \), and the PDE (6)-(7).

**Goal:** Show the coefficient \( q(x) \) is unique for each \( v^a(a + h, 2\tau) \), i.e. the map \( F : q(x) \mapsto v^a(a + h, 2\tau) \) is injective.

**Step 1.** Formulate the PDE as a difference of two solutions with the same boundary data, \( W^a = V^a_1 - V^a_2 \), where \( p = q_1 - q_2 \). Then derive an identity for \( W^a \).

**Step 2.** Derive an identity for a mean value operator of \( p, M(p) \) in terms of \( p \) and \( \nabla p \).

**Step 3.** Use steps 1 and 2 to estimate \( M(p) \) and \( \nabla M(p) \) in terms of \( p \) and \( \nabla p \), then use Gronwall’s to determine \( p = 0 \), i.e. \( q_1 = q_2 \).

Future Work and References
**Future Work:** A similar problem, but instead of working over the ellipsoid, \( |x - (a - h)| + |x - (a + h)| \leq 2\tau \), we work over the hyperboloid \( |x - a| - |x| \leq \tau \).

**References:**