An Eulerian-Lagrangian Runge-Kutta finite volume (ELRK-FV) method for solving convection-diffusion equations

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Introduction

We present a new Eulerian-Lagrangian (EL) finite volume method for solving the convection-diffusion equation,
\begin{equation}
\frac{du}{dt} + \nabla \cdot (u \phi) = \epsilon \Delta u + g(x,t), \quad x \in D, \quad t > 0.
\end{equation}

Standard Eulerian methods suffer from the CFL condition, resulting in needing to take small time steps for stability (CFL < 1). The EL framework loosens this constraint by tracing the characteristics backwards in time, that is, the traceback mesh moves (approximately) with the fluid velocity. This allows much larger time steps (CFL > 1) and reduces the computational cost.

Defining the space-time region $\Omega_j$

Consider the 1D case $u_t + f(u) = \epsilon u_{xx} + g(x,t)$. The particle velocities $u_{j+1}^n$ (i.e., slopes of the linear space-time curves $\tilde{z}_j(t)$) are defined using the Rankine-Hugoniot jump condition.

![Figure: The traceback space-time region $\Omega_j$ for cell $I_j(t)$](image)

The semi-discrete formulation

Integrating over $\Omega_j$ and applying the divergence theorem,
\begin{equation}
\frac{d}{dt} \int_{\Omega_j} u(x,t) dx = \int_{\Omega_j} \tilde{F}_j(t) - \tilde{F}_{j+1}(t) \quad \text{(A)}
\end{equation}
\begin{equation}
+ \epsilon \int_{\Omega_j} u_{xx}(x,t) dx + \int_{\Omega_j} q(x,t) dx \quad \text{(B)}
\end{equation}
\begin{equation}
\quad + \epsilon \int_{I_j} u_{xx}(x,t) dx + \int_{I_j} q(x,t) dx \quad \text{(C)}
\end{equation}
\begin{equation}
\quad + \epsilon \int_{I_j} u_{xx}(x,t) dx + \int_{I_j} q(x,t) dx \quad \text{(D)}
\end{equation}

where $F_j(t) \equiv f(u_j(t),t) - u_j(t)$ is the modified flux function. $F_{j+1}(t) = F_j(t) + u_j(t)$ is any monotone numerical flux.

Notation: Overlines denote uniform cell averages; tildes denote nonuniform cell averages.

Goal: Solve equation (2) using the method of lines.

The ELRK-FV algorithm

Step 1. Construct the approximate characteristics that are linear space-time curves, hence constructing $\Omega_j$ for $j = 1, 2, ..., N$.

Step 2. Compute the nonuniform cell averages $\{\tilde{u}_j(t^n) : j = 1, 2, ..., N\}$ at time $t^n$ using Remark 1 (below). These are the cell averages that we will evolve up to time $t^{n+1}$.

Step 3. Evolve the cell averages from $t^n$ to $t^{n+1}$ over $\Omega_j$. Use an IMEX Runge-Kutta scheme [1]. The terms (B), (C), and (D) on the right-hand-side of equation (2) can be evaluated as needed (below). Two-dimensional problems are solved with Strang splitting.

Evaluating terms (B), (C), and (D) in equation (2)

(B) We need the left and right limits $u_{j+1/2}^n$. Use local cell averages to compute WENO-AO [2] reconstruction polynomials, $\mathcal{R}_j(x \in I_j)$ for $j = 1, 2, ..., N$.

\begin{equation}
\text{Remark 1: nonuniform cell averages are computed by integrating } \mathcal{R}_j(x \in I_j) \text{ (using uniform cell averages) over each respective intersection of cells.}
\end{equation}

(C) The uniform cell averages $\bar{u}_{j+1/2}(t)$ are easily computed using the following equality:
\begin{equation}
\bar{u}_{j+1/2}(t) = D(t) \bar{u}_j(t),
\end{equation}

where D is a sparse Toeplitz matrix dependent on $\Delta x$ and the WENO-AO reconstruction polynomials. The desired nonuniform cell averages (C) are computed using Remark 1.

(D) Use a Gauss-Legendre quadrature of high enough order.

Numerical tests (the equilibrium solution)

The 0D2V $f = f(v_1, v_2, t)$ linearized Leonard-Bernstein Fokker-Planck equation is
\begin{equation}
f = \frac{1}{\epsilon} \nabla_v \cdot (v f) + \epsilon D \nabla_v^2 f,
\end{equation}

where $\epsilon = 1$, gas constant $R = 1/6$, temperature $T = 3$, number density $n = \pi$, and diffusion coefficient $D = RT/2$. The equilibrium solution is the Maxwellian,
\begin{equation}
f_{eq}(v_x, v_y) = \frac{n}{2\pi RT} \exp \left( -\frac{(v_x - \bar{v}_x)^2 + (v_y - \bar{v}_y)^2}{2RT} \right).
\end{equation}

Set $f = f_{eq}(v_1, v_2, t = 0) = f_{M1}(v_x, v_y) + f_{M2}(v_x, v_y)$, where $f_{M1}$ and $f_{M2}$ are randomly generated Maxwellians that preserve the macro-parameters.

![Figure: Numerical solution with $f_0 = f_{M1} + f_{M2}$. Mesh 200 x 200, CFL = 6. Times shown: 0.0, 0.3, 3.0.](image)

Numerical tests (order of convergence)

For testing convergence we set $f(v_1, v_2, t = 0) = f_{eq}(v_1, v_2)$ and use WENO-AO(5,3) (fourth-order in space due to diffusion) [2], IMEX(2,3,3) (third-order in time) [1], and Strang splitting (second-order in time).

![Figure: Left: convergence study with spatial mesh refinement at $T_f = 0.5$. Right: temporal convergence at $T_f = 0.1$.](image)

Takeaways:

- The spatial error dominates for small CFL numbers.
- The time-stepping and splitting errors dominate for larger CFL numbers.
- Very large CFL numbers and time steps are allowed!
- The method can handle nonlinear problems such as viscous Burgers’ equation (not shown).

Ongoing and future work

1. Modify the ELRK method to handle shocks/intersecting characteristics for hyperbolic conservation laws (ongoing).
2. Develop a non-splitting algorithm for two-dimensional problems.

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References