An Eulerian-Lagrangian Runge-Kutta finite volume (ELRK-FV) method for solving convection-diffusion equations

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Introduction

We present a new Eulerian-Lagrangian (EL) finite volume method for solving the convection-diffusion equation,

$$u_t + \nabla \cdot \mathbf{F}(u) = \epsilon \Delta u + g(\mathbf{x}, t), \quad \mathbf{x} \in \mathcal{D},\ t > 0. \tag{1}$$

Standard Eulerian methods suffer from the CFL condition, resulting in needing to take small time steps for stability ($CFL < 1$). The EL framework loosens this constraint by tracing the characteristics backwards in time, that is, the traceback mesh moves (approximately) with the fluid velocity. This allows much larger time steps ($CFL > 1$) and reduces the computational cost.

Defining the space-time region Ω_j

Consider the 1D case $u_t + f(u)_x = \epsilon u_{xx} + g(x,t)$. The particle velocities $\nu_{j+\frac{1}{2}}$ (i.e., slopes of the linear space-time curves $\tilde{x}_{j\pm\frac{1}{2}}(t)$) are defined using the Rankine-Hugoniot jump condition.

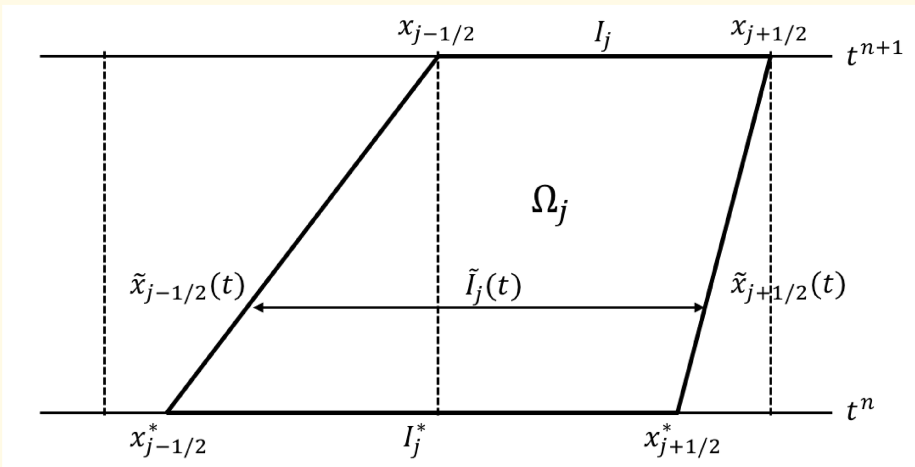

Figure: The traceback space-time region Ω_j for cell $\tilde{I}_j(t)$.

The semi-discrete formulation

Integrating over Ω_j and applying the divergence theorem,

$$\underbrace{\frac{d}{dt}\int_{\tilde{I}_j(t)} u(x,t)dx}_{\text{(A)}} = \underbrace{-\left[\hat{F}_{j+\frac{1}{2}}(t) - \hat{F}_{j-\frac{1}{2}}(t)\right]}_{\text{(B)}} + \epsilon \underbrace{\int_{\tilde{I}_j(t)} u_{xx}(x,t)dx}_{\text{(C)}} + \underbrace{\int_{\tilde{I}_j(t)} g(x,t)dx}_{\text{(D)}}, \tag{2}$$

where $F_{j+\frac{1}{2}}(t) \coloneqq f(u(\tilde{x}_{j+\frac{1}{2}}(t),t)) - \nu_{j+\frac{1}{2}} u(\tilde{x}_{j+\frac{1}{2}}(t),t)$ is the modified flux function. $\hat{F}_{j+\frac{1}{2}}(t) = \hat{F}_{j+\frac{1}{2}}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}; t)$ is any monotone numerical flux.

Notation: Overlines $\overline{u}_j$ denote uniform cell averages; tildes $\tilde{u}_j$ denote nonuniform cell averages.

Goal: Solve equation (2) using the method of lines.

The ELRK-FV algorithm

Step 1. Construct the approximate characteristics that are linear space-time curves, hence constructing Ω_j for $j = 1, 2, ..., N$.

Step 2. Compute the nonuniform cell averages $\{\tilde{u}_j(t^n) : j = 1, 2, ..., N\}$ at time t^n using Remark 1 (below). These are the cell averages that we will evolve up to time t^{n+1}.

Step 3. Evolve the cell averages from t^n to t^{n+1} over Ω_j. Use an IMEX Runge-Kutta scheme [1]. The terms (B), (C), and (D) on the righthand-side of equation (2) can be evaluated as needed (below). Two-dimensional problems are solved with Strang splitting.

Evaluating terms (B), (C), and (D) in equation (2)

(B) We need the left and right limits $u^{\pm}_{j+\frac{1}{2}}$. Use local cell averages to compute WENO-AO [2] reconstruction polynomials, $\mathcal{R}_j(x \in \tilde{I}_j)$ for $j = 1, 2, ..., N$.

$$u^-_{j+\frac{1}{2}} \approx \mathcal{R}_j(\tilde{x}_{j+\frac{1}{2}}(t)) \quad \text{and} \quad u^+_{j-\frac{1}{2}} \approx \mathcal{R}_j(\tilde{x}_{j-\frac{1}{2}}(t)) \tag{3}$$

Remark 1: nonuniform cell averages are computed by integrating $\mathcal{R}_j(x \in I_j)$ (using *uniform* cell averages) over each respective intersection of cells.

(C) The uniform cell averages $\overline{u}_{xx,j}(t)$ are easily computed using the following equation,

$$\overline{\mathbf{u}}_{xx}(t) = \mathbf{D}\overline{\mathbf{u}}(t), \tag{4}$$

where $\mathbf{D}$ is a sparse Toeplitz matrix dependent on Δx and the WENO-AO reconstruction polynomials. The desired nonuniform cell averages (C) are computed using Remark 1.

(D) Use a Gauss-Legendre quadrature of high enough order.

Numerical tests (the equilibrium solution)

The 0D2V ($f = f(v_x, v_y, t)$) linearized Leonard-Bernstein Fokker-Planck equation is

$$f_t = \frac{1}{\epsilon}\nabla_{\mathbf{v}} \cdot \left((\mathbf{v} - \overline{\mathbf{v}})f + D\nabla_{\mathbf{v}} f\right), \tag{5}$$

where $\epsilon = 1$, gas constant $R = 1/6$, temperature $T = 3$, number density $n = \pi$, and diffusion coefficient $D = RT = 1/2$. The equilibrium solution is the Maxwellian,

$$f_M(v_x, v_y) = \frac{n}{2\pi RT}\text{Exp}\left(-\frac{(v_x - \overline{v}_x)^2 + (v_y - \overline{v}_y)^2}{2RT}\right). \tag{6}$$

Set $f(v_x, v_y, t = 0) = f_{M1}(v_x, v_y) + f_{M2}(v_x, v_y)$, where f_{M1} and f_{M2} are randomly generated Maxwellians that preserve the macro-parameters.

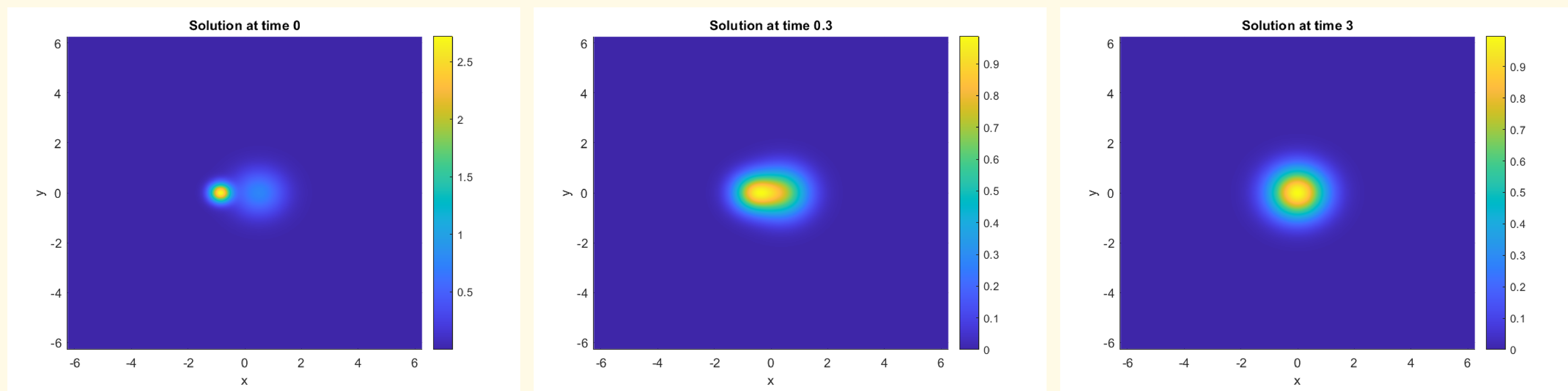

Figure: The numerical solution with $f_0 = f_{M1} + f_{M2}$. Mesh 200×200, $CFL = 6$. Times shown: 0.0, 0.3, 3.0.

Numerical tests (order of convergence)

For testing convergence we set $f(v_x, v_y, t = 0) = f_M(v_x, v_y)$ and use WENO-AO(5,3) (fourth-order in space due to diffusion) [2], IMEX(2,3,3) (third-order in time) [1], and Strang splitting (second-order in time).

$CFL = 0.95$		
$N_x = N_y$	L^1 Error	Order
50	9.07E-04	-
100	7.19E-05	3.66
200	5.35E-06	3.75
400	3.54E-07	3.92
$CFL = 8$		
$N_x = N_y$	L^1 Error	Order
50	5.70E-03	-
100	1.08E-03	2.40
200	1.69E-04	2.67
400	2.73E-05	2.63

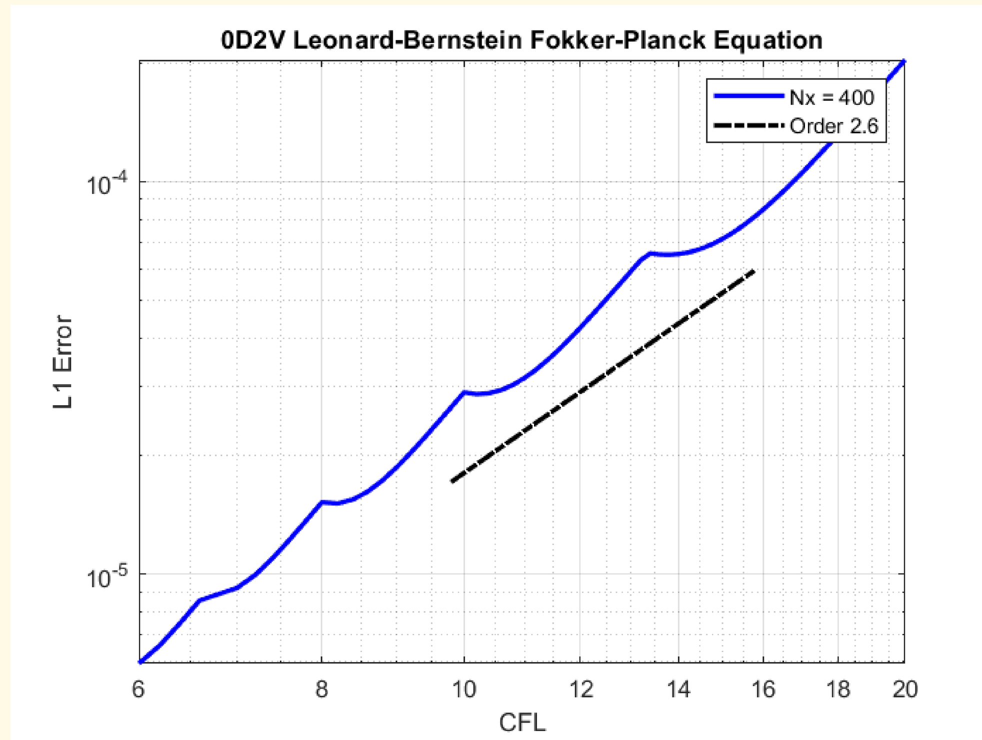

Figure: *Left:* convergence study with spatial mesh refinement at $T_f = 0.5$. *Right:* temporal convergence at $T_f = 0.1$.

Takeaways:

- The spatial error dominates for small CFL numbers.
- The time-stepping and splitting errors dominate for larger CFL numbers.
- Very large CFL numbers and time steps are allowed!
- The method can handle nonlinear problems such as viscous Burgers' equation (not shown).

Ongoing and future work

1. Modify the ELFV method to handle shocks/intersecting characteristics for hyperbolic conservation laws (ongoing).
2. Develop a non-splitting algorithm for two-dimensional problems.

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References

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[2] D.S. Balsara, S. Garain, and C.-W. Shu, An efficient class of WENO schemes with adaptive order, *J. Comput. Phys.*, **326** (2016), pp. 780-804.