Abstract

We present a semi-discrete HDG method for transient elastic waves that has a uniform-in-time superconvergence property. We show that the proof for superconvergence can be easily obtained by using a newly devised tailored projection and some existing techniques in traditional HDG methods. We also present numerical experiments that support our analysis. We finish with some simulations for elastic waves on a thick plate, using high order elements.

Model and numerical method

Geometry, $\Omega \in \mathbb{R}^3$, $\Gamma : = \partial \Omega$.

Transient elastic waves

\[ \rho \dot{u} = \nabla \sigma \cdot f \] (Newton’s law),

\[ A\sigma = \varepsilon(u) \] (Hook’s law),

\[ \varepsilon(u) = \frac{1}{3} \nabla u + \nabla u^T \] (Strain),

\[ u_{\Gamma} = g \] (Bd. condition).

HDG+ method. Find $\sigma_{h|K} \in P_h^{sym}$, $u_{h|K} \in P_{h+1}$, $a_{h|F} \in P_k$.

Projection and remainder

\[ \Pi : H^1(\Omega) \times H^1(\Omega) \to P_h^{sym}(K) \times P_{h+1}(K), \]

\[ (\sigma_{h|K}, u_{h|K} : = \Pi(\sigma, u; \Pi(\sigma, u)). \]

\[ R : H^1(\Omega) \times H^1(\Omega) \to R(K) := \prod_{F \in \mathcal{F}(K)} P_h(F), \]

\[ (\sigma_{h|K}, u_{h|K} \mapsto (\sigma_{h|K}, u_{h|K}). \]

If $\tau \in \mathcal{O}(h^k_{K})$ and $K$ shape regular, then

\[ \|\Pi(\sigma - \sigma_{h|K}) + h^2_{K}(|\Pi(\sigma - \sigma_{h|K})| + |h^2_{K}|) \leq C h_{K}^{k+1} \]

for $m = 1, 2, \ldots, k + 1$.

Devising HDG+ projection

\[ (P_h^{sym} + \Sigma_{h|K}) \times P_{h+1} \]

\[ M\text{-decomposition}[3, 2] \]

\[ (P_h^{sym} + \Sigma_{h|K}) \times P_{h+1} \]

\[ \text{weak commutativity} \]

\[ L^2 \text{ projection} \]

\[ P_h^{sym} \times P_{h+1} \]

\[ \text{boundary remainder} \]

Projection-based error analysis

Error terms

- Projected solution
  \[ \Pi(\sigma, u; \Pi(\sigma, u). \]

- Boundary remainder
  \[ \delta_{h|K} = \Pi(\sigma, u_K; \Pi(\sigma, u_K). \]

- Space discretization error
  \[ \varepsilon_h^k = \Pi(\sigma - \sigma_h, \varepsilon_h^k - \Pi u - u_h, \varepsilon_h^k - P_h u - u_h. \]

- Interpolation error
  \[ e_h = \Pi(\sigma, u_h). \]

Energy estimate

\[ \|\varepsilon_h^k\|_{L^2(\Omega)} \leq C \|e_h^k\|_{L^2(\Omega)} + \|e_h^k\|_{L^2(\Omega)} \]

Duality estimate (assume ellipticity regularity):

\[ \|\varepsilon_h^k\|_{L^2(\Omega)} \leq C \|e_h^k\|_{L^2(\Omega)} + \|e_h^k\|_{L^2(\Omega)} \]

\[ + \|e_h^k\|_{L^2(\Omega)} + \|e_h^k\|_{L^2(\Omega)} \]

\[ + \|e_h^k\|_{L^2(\Omega)} + \|e_h^k\|_{L^2(\Omega)}. \]

Simulation of elastic waves

Material parameters

- Geometry:
  \[ \Omega = [0, 1] \times [0, 1] \times [0, 0.05] \]

- Mass density:
  \[ \rho(x, y, z) = 1 + 2\{(x-0.5)(0.2)-(y-0.5)(0.2)-(z-0.5)(0.2) \}

- Lamé parameters:
  \[ \lambda(x, y, z) = 1, \mu(x, y, z) = 1. \]

Input of simulation data

- Vanishing forcing term: \[ f = 0. \]

- Neumann boundary with vanishing data (free surface):
  \[ y = 0, z = 0, \text{ and } z = 0.05. \]

- Dirichlet boundary: \[ x = 0, x = 1, \text{ and } y = 1. \]

- We apply an impulse of displacement in the $z$ direction.

Figure 1: Snapshots of elastic waves (shear waves) propagating in an inhomogeneous elastic plate. The time of snapshots are $t = 100/500, t = 260/500, t = 310/500, t = 330/500, t = 390/500, t = 490/500$. Space discretization by HDG+ with 2400 tetrahedrons and $k = 3$. Time discretization by trapezoidal rule Convolution Quadrature with 500 timesteps uniformly distributed in $[0, 1]$.

Figure 2: History of convergence for $\|\Pi u_h - \Pi u\|_{\Omega}$ and $\|u_h - u\|_{\Omega}$.

Numerical experiments

- Cubic domain with Dirichlet B.C.
- Isotropic elastic material.
- $C^0$ continuity exact solution
- HDG+ in space and Trapezoidal rule CQ in time
- $L^2$ error evaluated at fixed time $T$
- Space-time refinement with over-refinement in time

Conclusions & Future work

- HDG+ projection for linear elasticity with strong symmetric stress formulation.
- Projection-based error analysis proving optimal convergence of HDG+ in steady state linear elasticity and elastodynamics.
- Numerical experiments support our proof.
- Tailored projection for curl-curl formulation.
- Projection-based analysis for Maxwell equations.

References