HDG for transient elastic waves

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Abstract

We present a semi-discrete HDG method for transient elastic waves that has a uniform-in-time superconvergence property. We show that the proof for superconvergence can be easily obtained by using a newly devised tailored projection and some existing techniques in traditional HDG methods. We also present numerical experiments that support our analysis. We finish with some simulations for elastic waves on a thick plate, using high order elements.

Model and numerical method

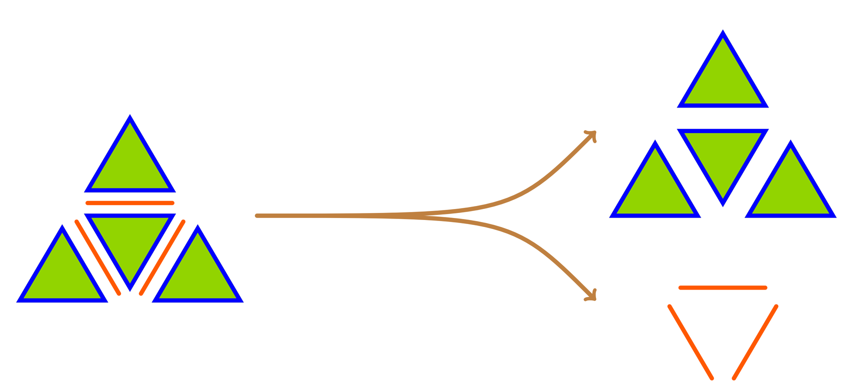
Geometry. $\Omega \subset \mathbb{R}^3$, $\Gamma := \partial\Omega$.

Transient elastic waves

$$\rho\ddot{\mathbf{u}} = \operatorname{div}\boldsymbol{\sigma} + \mathbf{f} \qquad \text{(Newton's law)},$$
$$\mathcal{A}\boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{(Hooke's law)},$$
$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\perp}) \qquad \text{(Strain)},$$
$$\mathbf{u}|_{\Gamma} = \mathbf{g} \qquad \text{(Bd. condition)}.$$

HDG+ method[1]. Find

$$\boldsymbol{\sigma}_h|_K \in \mathcal{P}_k^{\text{sym}}, \quad \mathbf{u}_h|_K \in \mathcal{P}_{k+1}, \quad \hat{\mathbf{u}}_h|_F \in \mathcal{P}_k.$$



- Local solver (given $\hat{\mathbf{u}}_h$, find $\boldsymbol{\sigma}_h, \mathbf{u}_h$):

$$(\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\theta})_{\mathcal{T}_h} + (\mathbf{u}_h, \operatorname{div}\boldsymbol{\theta})_{\mathcal{T}_h} - \langle\hat{\mathbf{u}}_h, \boldsymbol{\theta}\mathbf{n}\rangle_{\partial\mathcal{T}_h} = 0,$$
$$(\rho\ddot{\mathbf{u}}_h, \boldsymbol{w})_{\mathcal{T}_h} - (\operatorname{div}\boldsymbol{\sigma}_h, \boldsymbol{w})_{\mathcal{T}_h} + \langle\boldsymbol{\tau}\mathrm{P}_M(\mathbf{u}_h - \hat{\mathbf{u}}_h), \boldsymbol{w}\rangle_{\partial\mathcal{T}_h} = (\mathbf{f}, \boldsymbol{w})_{\mathcal{T}_h}.$$

- Global solver (solve a linear system about $\hat{\mathbf{u}}_h$):

$$\langle\widehat{\boldsymbol{\sigma}}_h\mathbf{n}, \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} := \langle\boldsymbol{\sigma}_h\mathbf{n} - \boldsymbol{\tau}\mathrm{P}_M(\mathbf{u}_h - \hat{\mathbf{u}}_h), \boldsymbol{\mu}\rangle_{\partial\mathcal{T}_h\setminus\Gamma} = 0.$$

HDG stabilization function $\boldsymbol{\tau}|_K \approx h_K^{-1}$.

Energy conservation. When there is no input data ($\mathbf{f}, \mathbf{g} = 0$), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\dot{\mathbf{u}}_h\|_\rho^2 + \|\boldsymbol{\sigma}_h\|_{\mathcal{A}} + |\mathrm{P}_M\mathbf{u}_h - \hat{\mathbf{u}}_h|_{\boldsymbol{\tau}}^2\right) = 0.$$

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HDG+ projection

Projection and remainder

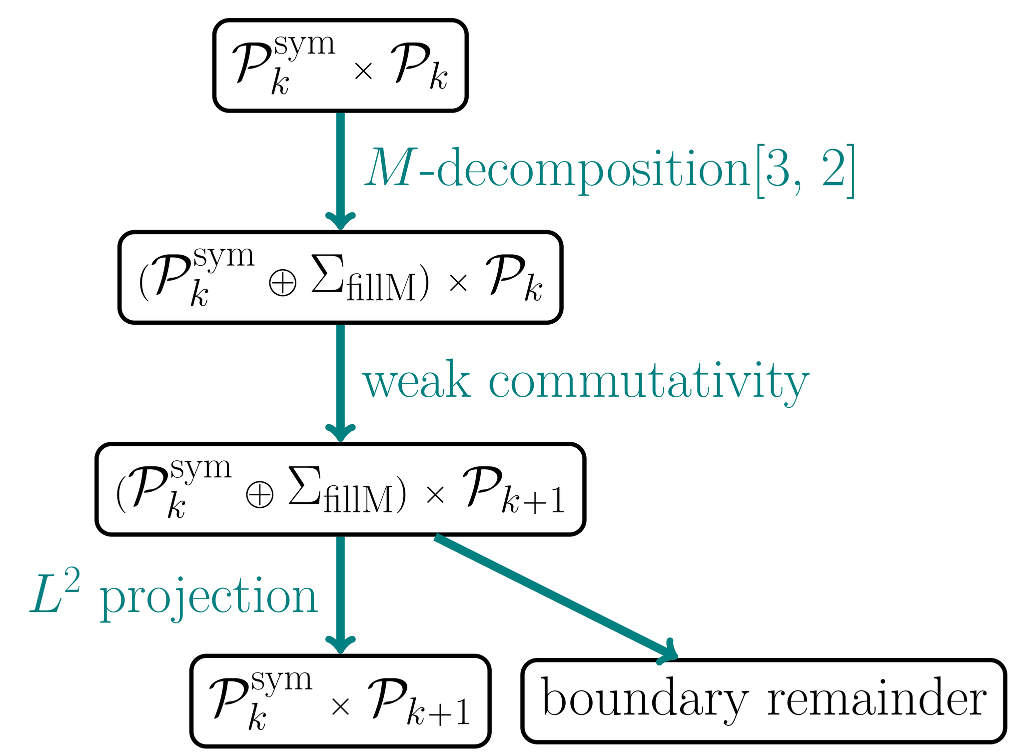
$$\Pi : H^1_{\text{sym}} \times H^1 \to \mathcal{P}_k^{\text{sym}}(K) \times \mathcal{P}_{k+1}(K),$$
$$(\boldsymbol{\sigma}, \mathbf{u}) \mapsto (\boldsymbol{\Pi}\boldsymbol{\sigma}, \boldsymbol{\Pi}\mathbf{u});$$
$$\mathrm{R} : H^1_{\text{sym}} \times H^1 \to \mathcal{R}_k(\partial K) := \prod_{F\in\varepsilon(K)} \mathcal{P}_k(F),$$
$$(\boldsymbol{\sigma}, \mathbf{u}) \mapsto \boldsymbol{\delta}.$$

If $\boldsymbol{\tau} = \mathcal{O}(h_K^{-1})$ and K shape regular, then

$$\|\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}\|_K + h_K^{-1}\|\Pi\mathbf{u} - \mathbf{u}\|_K + h_K^{1/2}\|\boldsymbol{\delta}\|_{\partial K} \leqslant Ch_K^m(|\boldsymbol{\sigma}|_{m,K} + |\mathbf{u}|_{m+1,K}),$$

for $m = 1, 2, ..., k+1$.

Devising HDG+ projection



Projection-based error analysis

Error terms

- Projected solution
  $(\boldsymbol{\Pi}\boldsymbol{\sigma}, \boldsymbol{\Pi}\mathbf{u}) := \prod_{K\in\mathcal{T}_h}\boldsymbol{\Pi}(\boldsymbol{\sigma}|_K, \mathbf{u}|_K; \boldsymbol{\tau}|_K)$
- Boundary remainder
  $\boldsymbol{\delta} := \prod_{K\in\mathcal{T}_h}\mathbf{R}(\boldsymbol{\sigma}|_K, \mathbf{u}|_K; \boldsymbol{\tau}|_K)$
- Space discretization error
  $\boldsymbol{\varepsilon}_h^\sigma := \boldsymbol{\Pi}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$, $\boldsymbol{\varepsilon}_h^u := \boldsymbol{\Pi}\mathbf{u} - \mathbf{u}_h$, $\hat{\boldsymbol{\varepsilon}}_h^u := \mathrm{P}_M\mathbf{u} - \hat{\mathbf{u}}_h$
- Interpolation error
  $\mathbf{e}_\sigma := \boldsymbol{\Pi}\boldsymbol{\sigma} - \boldsymbol{\sigma}$, $\mathbf{e}_u := \boldsymbol{\Pi}\mathbf{u} - \mathbf{u}$

Energy estimate

$$\|\boldsymbol{\varepsilon}_h^\sigma\|_{\mathcal{A}}^{L^\infty_{[0,T]}} + \|\mathrm{P}_M\boldsymbol{\varepsilon}_h^u - \boldsymbol{\varepsilon}_h^{\widehat{u}}\|_{\boldsymbol{\tau}}^{L^\infty_{[0,T]}} + \|\dot{\boldsymbol{\varepsilon}}_h^u\|_\rho^{L^\infty_{[0,T]}} \leqslant C\Big(\|\boldsymbol{e}_\sigma(0)\|_{\mathcal{A}} + \|\dot{\boldsymbol{e}}_\sigma\|_{\mathcal{A}}^{L^1_{[0,T]}} + \|\ddot{\boldsymbol{e}}_u\|_\rho^{L^1_{[0,T]}} + \|\boldsymbol{\delta}\|_{\boldsymbol{\tau}^{-1}}^{L^\infty_{[0,T]}} + \|\dot{\boldsymbol{\delta}}\|_{\boldsymbol{\tau}^{-1}}^{L^1_{[0,T]}}\Big).$$

Duality estimate (assume elliptic regularity):

$$\|\boldsymbol{\varepsilon}_h^u\|_\Omega^{L^\infty_{[0,T]}} \leqslant C(1+T)^2\Big(h\|\mathbf{e}_\sigma(0)\|_\Omega + h\|\boldsymbol{\delta}(0)\|_{\boldsymbol{\tau}^{-1}} + \|\mathbf{e}_u(0)\|_\Omega + h\|\dot{\mathbf{e}}_\sigma\|_\Omega^{L^\infty_{[0,T]}} + h\|\dot{\boldsymbol{\delta}}\|_{\boldsymbol{\tau}^{-1}}^{L^\infty_{[0,T]}} + \|\ddot{\mathbf{e}}_u\|_\Omega^{L^\infty_{[0,T]}} + h\|\ddot{\mathbf{e}}_\sigma\|_\Omega^{L^\infty_{[0,T]}} + h\|\ddot{\boldsymbol{\delta}}\|_{\boldsymbol{\tau}^{-1}}^{L^\infty_{[0,T]}} + \|\dddot{\mathbf{e}}_u\|_\Omega^{L^\infty_{[0,T]}}\Big).$$

Simulation of elastic waves

Material parameters

- Geometry:
  $\Omega = [0,1] \times [0,1] \times [0,0.05]$
- Mass density:
  $\rho(x,y,z) = 1 + 2\chi_{|x-0.5|\leqslant 0.2 \cap |y-0.5|\leqslant 0.2 \cap |z-0.5|\leqslant 0.2}$.
- Lamé parameters:
  $\lambda(x,y,z) \equiv 1$, $\mu(x,y,z) \equiv 1$.

Input of simulation data

- Vanishing forcing term: $\mathbf{f} \equiv 0$.
- Neumann boundary with vanishing data (free surface): $y = 0$, $z = 0$, and $z = 0.05$.
- Dirichlet boundary: $x = 0$, $x = 1$, and $y = 1$. We apply an impulse of displacement in the z direction.

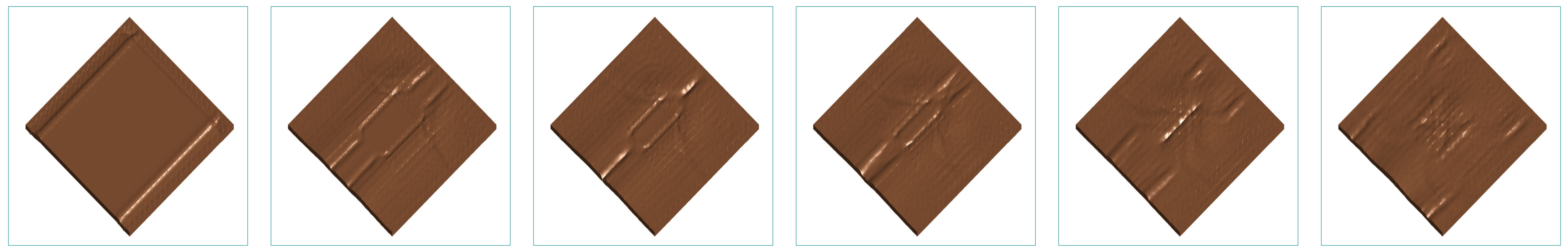


Figure 1: Snapshots of elastic waves (shear waves) propagating in an inhomogeneous elastic plate. The time of snapshots are $t = 100/500$, $t = 260/500$, $t = 310/500$, $t = 330/500$, $t = 390/500$, $t = 490/500$. Space disretization by HDG+ with 2400 tetrahedras and $k = 3$; Time discretization by trapezoidal rule Convolution Quadrature with 500 timesteps uniformly distributed in $[0,1]$.

Numerical experiments

- Cubic domain with Dirichlet B.C.
- Isotropic elastic material.
- $C^\infty(\overline{\Omega})$ exact solution
- HDG+ in space and Trapezoidal rule CQ in time
- L^2 error evaluated at fixed time T
- Space-time refinement with over-refinement in time

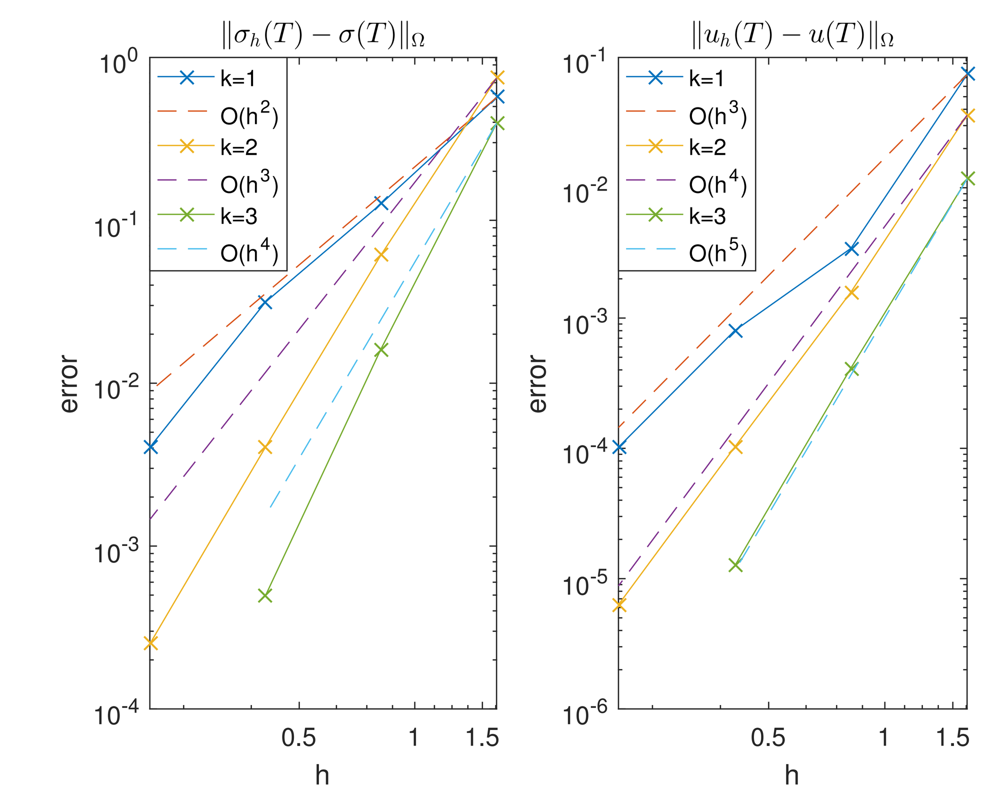


Figure 2: History of convergence for $\boldsymbol{\sigma}(T)$ and $\boldsymbol{u}(T)$.

Conclusions & Future work

- HDG+ projection for linear elasticity with strong symmetric stress formulation. ✓
- Projection-based error analysis proving optimal convergence of HDG+ in steady state linear elasticity and elastodynamics. ✓
- Numerical experiments support our proof. ✓
- Tailored projection for curl-curl formulation.
- Projection-based analysis for Maxwell equations.

References

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