BIE for scattering of transient waves by homogeneous penetrable obstacles

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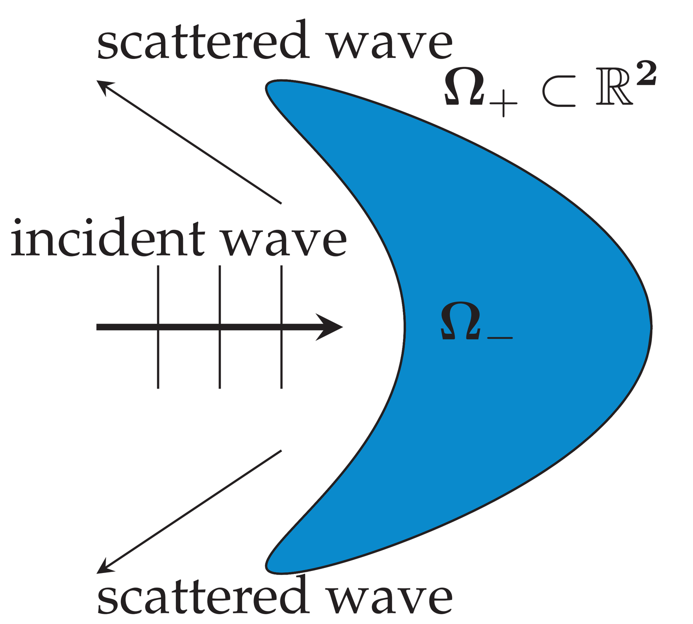
Abstract

We study the scattering of acoustic waves in free space when hitting bounded obstacles with different material properties. The problem can be reformulated using a time-domain version of the Costabel-Stephan system of BIE, using two unknowns on the interfaces and the entire collection of operators from the Calderon projector. We show that there is a well posed (stable in time) dynamical system that describes the system of integral equation, and that well posedness is not affected by Galerkin semidiscretization. Finally, we derive error estimates for a full discretization of the problem, using Galerkin methods in space and Convolution Quadrature in time.

BIE for transmission problem

Transmission Problem

$$\begin{aligned}
&c^{-2}\ddot{u}(t) = \alpha\Delta u(t) && \text{in } \Omega_-,\\
&\gamma^- u(t) - \gamma^+ u(t) = \gamma u_{in}(t) && \text{on } \Gamma,\\
&\alpha\partial_\nu^- u - \partial_\nu^+ u = \partial_\nu u_{in}(t) && \text{on } \Gamma,\\
&\ddot{u}(t) = \Delta u(t) && \text{in } \Omega_+,\\
&u(0) = \dot{u}(0) = 0.
\end{aligned}$$



Choose two unknowns to be

$$\lambda(t) := \partial_\nu^- u(t), \quad \phi(t) := \gamma^- u(t).$$

The solution can be represented from the boundary as

$$u = \begin{cases} \mathcal{S}_{c\sqrt{\alpha}} * \lambda - \mathcal{D}_{c\sqrt{\alpha}} * \phi & \text{in } \Omega_-,\\ -\mathcal{S} * (\alpha\lambda - \partial_\nu u_{in}) + \mathcal{D} * (\phi - \gamma u_{in}) & \text{in } \Omega_+.\end{cases}$$

Plug this into the boundary conditions and utilize the jump conditions of potentials to obtain the Costabel-Stephan BIE system

$$\begin{bmatrix} \mathcal{V}_{c\sqrt{\alpha}} + \alpha\mathcal{V} & -\mathcal{K}_{c\sqrt{\alpha}} - \mathcal{K} \\ \mathcal{J}_{c\sqrt{\alpha}} + \mathcal{J} & \mathcal{W}_{c\sqrt{\alpha}} + \alpha^{-1}\mathcal{W} \end{bmatrix} * \begin{bmatrix} \lambda \\ \phi \end{bmatrix} = \begin{bmatrix} \mathcal{V} & \frac{1}{2}I - \mathcal{K} \\ \alpha^{-1}(\frac{1}{2}I + \mathcal{J}) & \alpha^{-1}\mathcal{W} \end{bmatrix} * \begin{bmatrix} \partial_\nu u_{in} \\ \gamma u_{in} \end{bmatrix}$$

Finally, the problem is solved by substituting λ, ϕ back into the boundary representation.

Discretization in time and space

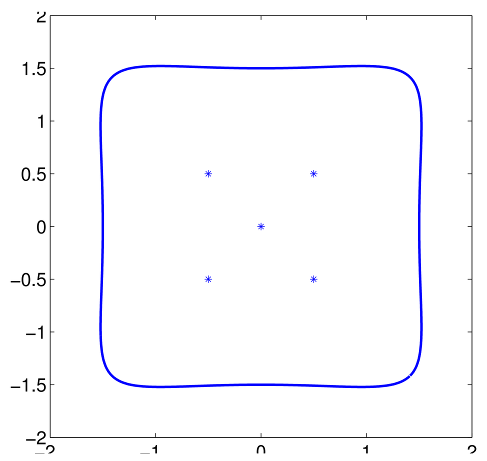
In time, we discretize by BDF2 based Convolution Quadrature.
In space, we use Galerkin semidiscretization which requires two sequences of closed subspaces $X_h \subset H^{-1/2}(\Gamma)$, $Y_h \subset H^{1/2}(\Gamma)$. Find $\lambda_h : \mathbb{R} \to X_h, \phi_h : \mathbb{R} \to Y_h$ such that

$$\begin{aligned}
&\langle \mu_h, (\mathcal{V}_{c\sqrt{\alpha}} + \alpha\mathcal{V}) * \lambda_h\rangle - \langle \mu_h, (\mathcal{K}_{c\sqrt{\alpha}} + \mathcal{K}) * \phi_h\rangle \\
&\quad = \langle \mu_h, \mathcal{V} * \partial_\nu u_{in}\rangle + \langle \mu_h, (\tfrac{1}{2}I - \mathcal{K}) * \gamma u_{in}\rangle \qquad \forall \mu_h \in X_h,\\
&\langle (\mathcal{J}_{c\sqrt{\alpha}} + \mathcal{J}) * \lambda_h, \varphi_h\rangle + \langle (\mathcal{W}_{c\sqrt{\alpha}} + \alpha^{-1}\mathcal{W}) * \phi_h, \varphi_h\rangle \\
&\quad = \langle \alpha^{-1}(\tfrac{1}{2}I + \mathcal{J}) * \partial_\nu u_{in}, \varphi_h\rangle + \langle \alpha^{-1}\mathcal{W} * \gamma u_{in}, \varphi_h\rangle \qquad \forall \varphi_h \in Y_h.
\end{aligned}$$

Finally, u_h is computed from λ_h and φ_h using the integral formula.

$$u_h = \begin{cases} \mathcal{S}_{c\sqrt{\alpha}} * \lambda_h - \mathcal{D}_{c\sqrt{\alpha}} * \phi_h & \text{in } \Omega_-,\\ -\mathcal{S} * (\alpha\lambda_h - \partial_\nu u_{in}) + \mathcal{D} * (\phi_h - \gamma u_{in}) & \text{in } \Omega_+.\end{cases}$$

Numerical Experiment 1



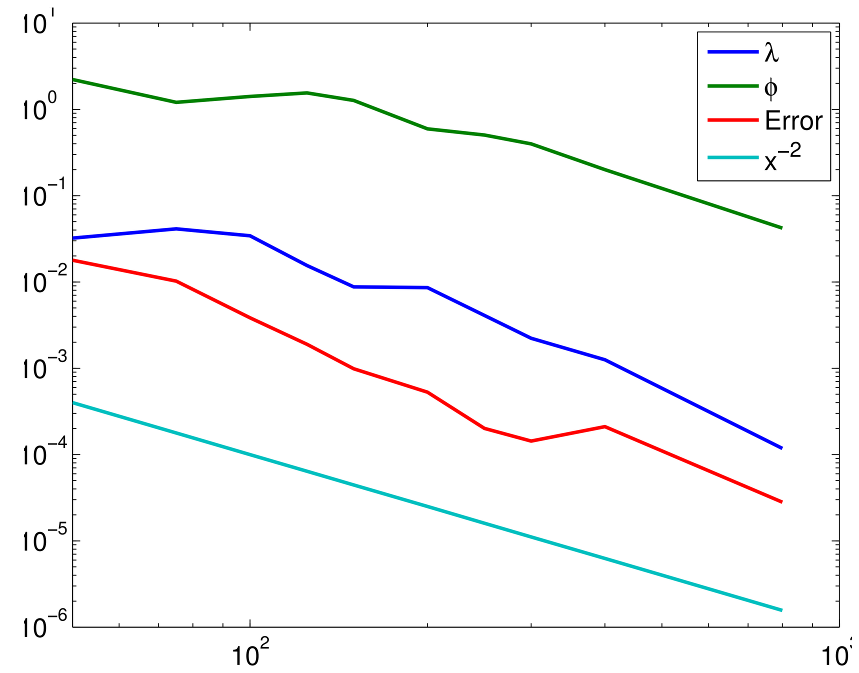
The domain in the numerical experiment is shown on the left. The incident wave is a plane wave heading in the SE direction:

$$u_{in} = \sin(ct - \mathbf{x}\cdot\mathbf{d})H(ct - \mathbf{x}\cdot\mathbf{d}), \mathbf{d} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

where $H(y) = \left(\frac{-2}{k^3}y^3 + \frac{3}{k^2}y^2\right)\chi_{y<k} + \chi_{y>k}$ is intended for the data to take off smoothly. The constants $c = 1, \alpha = 1, k = 0.5$. The exact solution to the problem is

$$\lambda = \partial_\nu u_{in}, \quad \phi = \gamma u_{in}, \quad u|_{\Omega_+} \equiv 0, \quad u|_{\Omega_-} = u_{in}$$

We used the deltaBEM to get the solution up till time $T = 5$. The error of λ, ϕ and the solution on five points(marked on the graph above) inside the domain are checked at the last time step. The graph on the right shows that the error decreases at order two.



Numerical Experiment 2

In the second experiment, the kite shaped object is hit by a plane wave

$$u_{in} = \sin(ct - \mathbf{x}\cdot\mathbf{d})\,H(ct - \mathbf{x}\cdot\mathbf{d}), \quad \mathbf{d} = (1, 0)$$

The constants $c = 2/3, \alpha = 3/2, k = 0.5$. The following pictures are again computed using the deltaBEM method. We can clearly see the exterior wave goes a little faster than the interior wave.

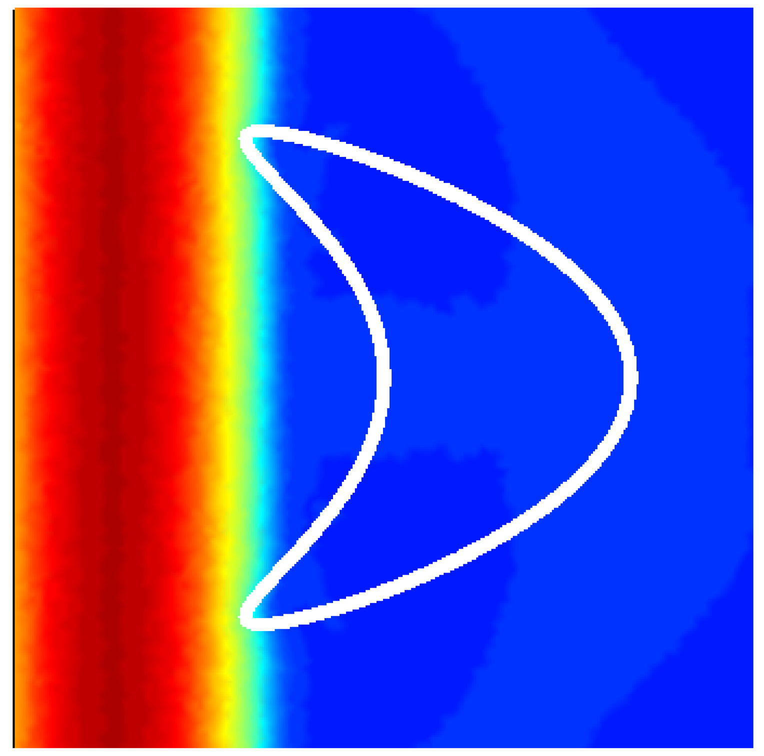


Figure 1: t=0

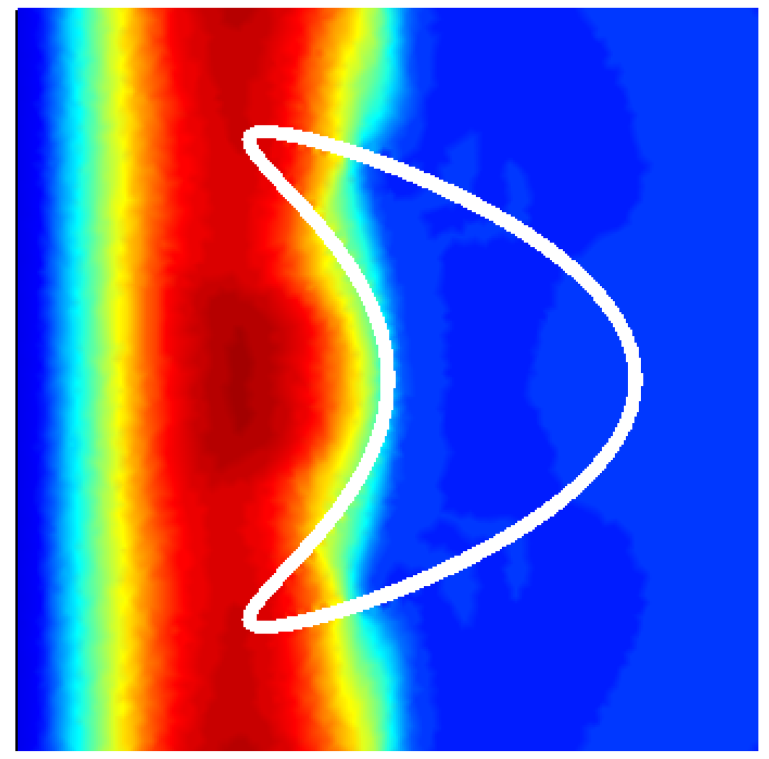


Figure 2: t=1

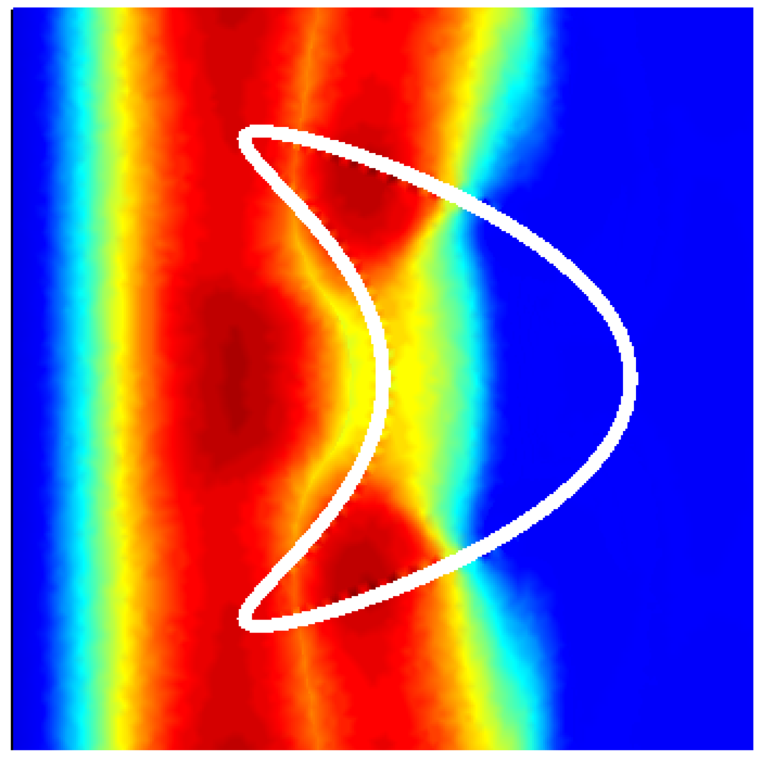


Figure 3: t=2.25

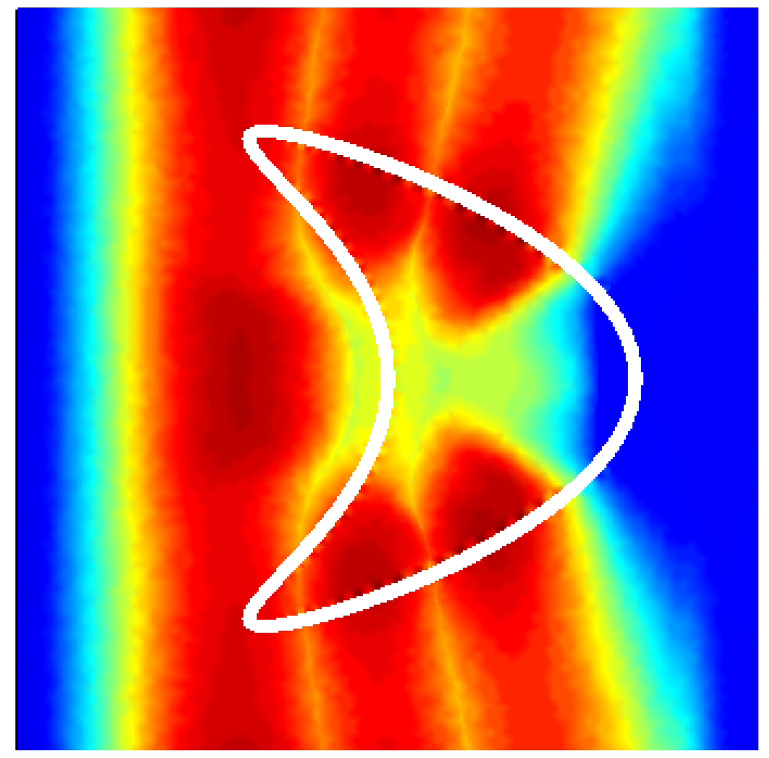


Figure 4: t=5

Theoretical Error Estimate

Denote the error as $\epsilon_h = (u_h - u)|_{\Omega_-}$ and introduce the space

$$W_+^k(\mathbb{R}; X) := \left\{\beta \in C^{k-1}(\mathbb{R}; X) : \mathrm{supp}\beta \subset [0, \infty), \beta^{(k)} \in L^1_{loc}(\mathbb{R}; X)\right\}$$

Proposition 1. *If $\phi \in W_+^2(\mathbb{R}, H^{1/2}(\Gamma)), \lambda \in W_+^2(\mathbb{R}, H^{-1/2}(\Gamma))$, then*

$$\epsilon_h \in C_+^1(\mathbb{R}; L^2(\mathbb{R}^d)) \cap C_+(\mathbb{R}; H^1(\mathbb{R}^d \backslash \Gamma)) \quad \textit{and}$$

$$\|\epsilon_h(t)\|_{1,\mathbb{R}^d} \le C\left(H_3(\partial^{-1}\phi, t|H^{1/2}(\Gamma)) + H_3(\partial^{-1}\lambda, t|H^{-1/2}(\Gamma))\right)$$

where

$$H_k(\varphi, t|X) := \sum_{l=0}^{k} \int_0^t \|\varphi^{(l)}(\tau)\|_X \, d\tau$$

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