ABSTRACT
We study the scattering of acoustic waves in free space when hitting bounded obstacles with different material properties. The problem can be reformulated using a time-domain version of the Costabel-Stephan system of BIE, using two unknowns on the interfaces and the entire collection of operators from the Calderon projector. We show that there is a well posed (stable in time) dynamical system that describes the system of integral equations, and that well posedness is not affected by Galerkin semidiscretization. Finally, we derive error estimates for a full discretization of the problem, using Galerkin methods in space and Convolution Quadrature in time.

BIE FOR TRANSMISSION PROBLEM
Transmission Problem

\[ c^2 \ddot{u}(t) = \alpha \Delta u(t) \quad \text{in} \quad \Omega_+ \subset \mathbb{R}^2 \]
\[ \gamma^- u(t) - \gamma^+ u(t) = \gamma u_{in}(t) \quad \text{on} \quad \Gamma, \]
\[ \alpha \ddot{u} - \alpha \dot{\gamma} u + \partial_{\nu} u_{in}(t) \quad \text{on} \quad \Gamma, \]
\[ \dot{u}(t) = \Delta u(t) \quad \text{in} \quad \Omega_+, \]
\[ u(0) = u(0) = 0. \]

Choose two unknowns to be
\[ \lambda(t) := \partial_{\nu} u(t), \quad \phi(t) := \gamma^- u(t). \]

The solution can be represented from the boundary as
\[ u = \begin{cases} S_{\nu,\gamma} \lambda - D_{\nu,\gamma} \phi & \text{in} \quad \Omega_+, \\ -S \cdot (\alpha \lambda - \partial_{\nu} u_{in}) + D \cdot (\phi - \gamma u_{in}) & \text{in} \quad \Omega_. \end{cases} \]

Plug this into the boundary conditions and utilize the jump conditions of potentials to obtain the Costabel-Stephan BIE system
\[ \begin{bmatrix} V_{\nu,\gamma} + \alpha V \\ J_{\nu,\gamma} + J \end{bmatrix} \begin{bmatrix} \lambda \\ \phi \end{bmatrix} = \begin{bmatrix} \alpha \lambda + \partial_{\nu} u_{in} \\ \gamma u_{in} \end{bmatrix}. \]

Finally, the problem is solved by substituting \( \lambda, \phi \) back into the boundary representation.

DISCRETIZATION IN TIME AND SPACE
In time, we discretize by BD2 based Convolution Quadrature.

\[ u_{in} = \sin(c t - x \cdot d) H(ct - x \cdot d), \quad d = (1, 0) \]

The constants \( c = 2/3, \alpha = 3/2, k = 0.5 \). The following pictures are again computed using the deltaBEM method. We can clearly see the exterior wave goes a little faster than the interior wave.

NUMERICAL EXPERIMENT 1

The domain in the numerical experiment is shown on the left. The incident wave is a plane wave heading in the SE direction: \( u_{in} = \sin(ct - x \cdot d) H(ct - x \cdot d) \).

\[ u_{in} = \begin{cases} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot \chi_{x<0} \cdot \chi_{y<0} & \text{int} \quad \Omega, \\ 0 & \text{on} \quad \Gamma. \end{cases} \]

The solution is computed using the deltaBEM method. We can clearly see the scattered wave on the right shows that the error decreases at order two.

NUMERICAL EXPERIMENT 2

In the second experiment, the kite shaped object is hit by a plane wave

\[ u_{in} = \sin(c t - x \cdot d) H(ct - x \cdot d), \quad d = (1, 0) \]

The constants \( c = 2/3, \alpha = 3/2, k = 0.5 \). The following pictures are again computed using the deltaBEM method. We can clearly see the exterior wave goes a little faster than the interior wave.

THEORETICAL ERROR ESTIMATE
Denote the error as \( \epsilon_h = (u_h - u)|_{\Omega_+} \) and introduce the space

\[ W^2 \left( \mathbb{R}; X \right) := \{ \beta \in C^{k-1} \left( \mathbb{R}; X \right) \mid \text{supp} \beta \subset [0, \infty), \beta^{(k)} \in L^1_{loc}(\mathbb{R}; X) \} \]

Proposition 1. If \( \phi \in W^2 \left( \mathbb{R}, H^{1/2}(\Gamma) \right), \lambda \in W^2 \left( \mathbb{R}, H^{-1/2}(\Gamma) \right) \), then

\[ \epsilon_h \in C^0 \left( \mathbb{R}; L^2(\mathbb{R}^d) \right) \cap C_+ \left( \mathbb{R}; H^1(\mathbb{R}^d; \Gamma) \right) \]

and

\[ \| \epsilon_h(t) \|_{1, \Gamma, 0} \leq C \left( H_0(\delta^{-1} \lambda, t; H^{1/2}(\Gamma)) + H_0(\delta^{-1} \phi, t; H^{-1/2}(\Gamma)) \right) \]

where

\[ H_0(\phi, t|X) := \sum_{l=0}^k \int_0^t \| \phi^{(l)}(\tau) \|_X d\tau \]

REFERENCES