EDGE-DISJOINT SPANNING TREES

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ABSTRACT

Partially answering a question of Seymour, we obtain a sufficient eigenvalue condition for the existence of $k$ edge-disjoint spanning trees in a regular graph, when $k \in \{2, 3\}$. We construct examples of graphs that show our bounds are essentially best possible.

STANDARD THEORY

A graph $G$ is an ordered pair $(V, E)$, where $V$ is a set of vertices and $E$ is a set of edges. The number of edges coming out of a vertex $v$ is called the degree, and the graph is regular if every vertex has the same degree $d$. A tree is a connected graph with no cycles, and spanning if its edges are a subset of the edges of $G$, and it contains all vertices of $G$.

Kirchhoff's Matrix Tree Theorem [3] is a classic result relating eigenvalues and spanning trees, given as follows.

Theorem 1. Let $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of $L(G)$. Then the number of spanning trees of $G$ is $\prod_{i=1}^{n} \frac{\mu_i}{\mu_1}$.

Given two square matrices $A$ and $B$, with dimensions $n$ and $m$ respectively, $m \geq n$, the eigenvalues of $B$ interface those of $A$ if for $1 \leq i \leq m$, $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A)$, where $\lambda_i(A)$ is the $i$-th largest eigenvalue of $A$ [1].

The eigenvalues of $A(H)$ interlace those of $A(G)$, where $A(H)$ is a principal submatrix of $A(G)$. $A(H)$ can be seen as its own adjacency matrix and thus is represented by a graph $H$, which is called an induced subgraph of $G$.

If the vertex set is partitioned in $t$ parts, the quotient matrix of a graph is a $t \times t$ matrix whose $(i, j)$ entry is the average number of edges going between part $i$ to part $j$. The eigenvalues of a quotient matrix interlace the eigenvalues of $A(G)$ [1].

A sufficient and necessary condition for edge-disjoint spanning trees was proven by Nash-Williams and Tutte independently ([4], [5]), which is stated as follows:

Theorem 2. Let $\sigma(G)$ denote the maximum number of edge-disjoint spanning trees of $G$. Then $\sigma(G) \geq k$ if and only if for all partitions of the vertex set into $t$ parts, the number of edges joining amongst the $t$ parts is at least $k(t-1)$.

We provide a sufficient eigenvalue condition for a graph to have at least 2 and 3 edge-disjoint spanning trees. These structures are important in computer science and chemistry.

EXTREMAL CONFIGURATIONS

The graph to the far left is a 5-regular graph containing 1 edge-disjoint spanning tree and $\lambda_2 \approx 4.62 > 5 - \frac{3}{5} \approx 4.80$. This shows that there are graphs with $\lambda_2$ slightly above the bound that fail the conclusion of Theorem 2.

The other graph is a 10-regular graph containing 2 edge-disjoint spanning trees and $\lambda_3 \approx 9.609 > 10 - \frac{5}{10} \approx 9.50$. This shows that there are graphs with $\lambda_3$ slightly above the bound that fail the conclusion of Theorem 3.

RESULTS

These are our results.

Theorem 3. Let $d \geq 4$ and $G$ be a $d$-regular graph such that $\lambda_2 < d - \frac{3}{d-1}$. Then $G$ contains at least 2 edge-disjoint spanning trees.

Theorem 4. Let $d \geq 6$ and $G$ be a $d$-regular graph such that $\lambda_2 < d - \frac{3}{d-1}$. Then $G$ contains at least 3 edge-disjoint spanning trees.

We constructed the extremal configurations below, which shows the bounds are best possible. To find a similar sufficient eigenvalue condition for more than 3 edge-disjoint spanning trees remains open due to the large increase in cases to consider and from using other results that are only known for small values. We conjecture the following.

Conjecture 5. Let $d \geq 8$ and $4 \leq k \leq \lfloor \frac{d}{2} \rfloor$ be two integers. If $G$ is a $d$-regular graph such that $\lambda_2(G) < d - \frac{2k}{d-2k}$, then $\sigma(G) \geq k$.

CASES OF THEOREM 3

The proof of both theorems use the contrapositive. In Theorem 3, we assume a graph has exactly 1 edge disjoint spanning tree. Then by Theorem 2, there exists a partition of the vertex set with less than 2$(t-1)$ edges going amongst the parts. The restriction on the edges narrows down the possible structures of the graph. A possible example is

CASES OF THEOREM 4

In Theorem 4, the contrapositive assumes there exists a partition of the vertex set with less than 3$(t-1)$ edges going amongst the parts. Two possible structures of a graph are given below.

REFERENCES