Edge-Disjoint Spanning Trees

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Abstract

Partially answering a question of Seymour, we obtain a sufficient eigenvalue condition for the existence of k edge-disjoint spanning trees in a regular graph, when $k \in \{2, 3\}$. We construct examples of graphs that show our bounds are essentially best possible.

Standard Theory

A **graph** G is an ordered pair (V, E), where V is a set of vertices and E is a set of edges. The number of edges coming out of a vertex v is called the **degree**, and the graph is **regular** if every vertex has the same degree d. A **tree** is a connected graph with no cycles, and **spanning** if its edges are a subset of the edges of G, and it contains all vertices of G.

If $|V| = n$, then its **adjacency matrix** $A(G)$ is an $n \times n$ matrix where its (i, j) entry is 1 if v_i is adjacent to v_j, and 0 otherwise. The **Laplacian Matrix** of a graph G is $L(G) = D - A(G)$, where D is a diagonal matrix whose i-th diagonal entry is the degree of vertex i.

Kirchhoff's Matrix Tree Theorem [3] is a classic result relating eigenvalues and spanning trees, given as follows.

Theorem 1. *Let $0 = \mu_1 \le \mu_2 \le \ldots \le \mu_n$ be the eigenvalues of $L(G)$. Then the number of spanning trees of G is $\frac{\prod_{i=2}^{n} \mu_i}{n}$.*

Given two square matrices A and B, with dimensions n and m respectively, $n \ge m$, the eigenvalues of B **interlace** those of A if for $1 \le i \le m$,

$$\lambda_i(A) \ge \lambda_i(B) \ge \lambda_{n-m+i}(A),$$

where $\lambda_i(A)$ is the i-th largest eigenvalue of A [1]. The eigenvalues of $A(H)$ interlace those of $A(G)$, where $A(H)$ is a principal submatrix of $A(G)$. $A(H)$ can be seen as its own adjacency matrix and thus is represented by a graph H, which is called an **induced subgraph** of G.

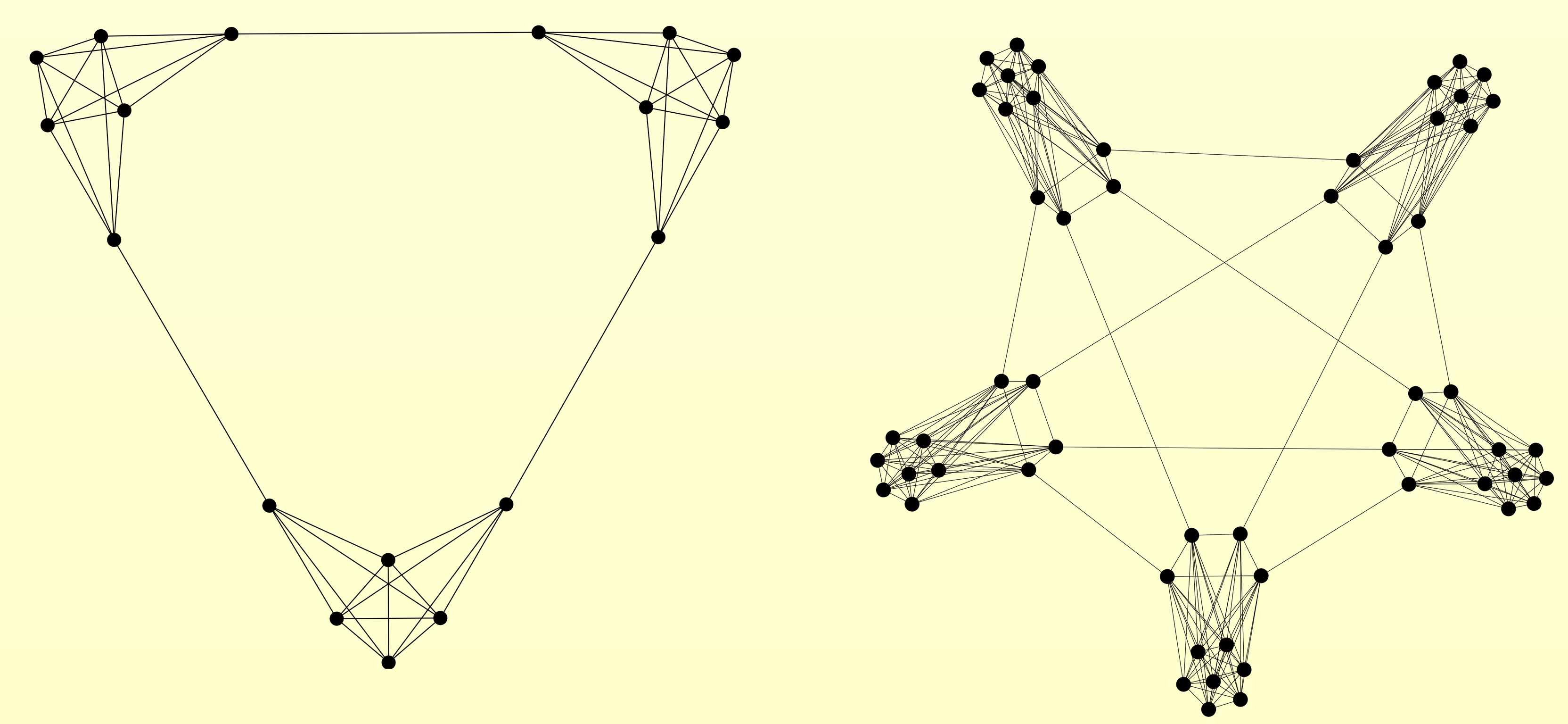
If the vertex set is partitioned in t parts, the **quotient matrix** of a graph is a $t \times t$ matrix whose (i, j) entry is the *average* number of edges going between part i to part j. The eigenvalues of a quotient matrix interlace the eigenvalues of $A(G)$ [1] .

A sufficient and necessary condition for edge-disjoint spanning trees was proven by Nash-Williams and Tutte independently ([4], [5]), which is stated as follows:

Theorem 2. *Let $\sigma(G)$ denote the maximum number of edge-disjoint spanning trees of G. Then $\sigma(G) \ge k$ if and only if for all partitions of the vertex set into t parts, the number of edges joining amongst the t parts is* at least $k(t-1)$.

We provide a sufficient eigenvalue condition for a graph to have at least 2 and 3 edge-disjoint spanning trees. These structures are important in computer science and chemistry.

Extremal Configurations



The graph to the far left is a 5-regular graph containing 1 edge-disjoint spanning tree and $\lambda_2 \approx 4.62 > 5 - \frac{3}{5+2} \approx 4.57$. This shows that there are graphs with λ_2 slightly above the bound that fail the conclusion of Theorem 2.

The other graph is a 10-regular graph containing 2 edge-disjoint spanning trees and $\lambda_2 \approx 9.609 > 10 - \frac{5}{10+2} \approx 9.583$. This shows that there are graphs with λ_2 slightly above the bound that fail the conclusion of Theorem 3.

Results

These are our results.

Theorem 3. *Let $d \ge 4$ and G be a d-regular graph such that $\lambda_2 < d - \frac{3}{d+1}$. Then G contains at least 2-edge disjoint spanning trees.*

Theorem 4. *Let $d \ge 6$ and G be a d-regular graph such that $\lambda_2 < d - \frac{5}{d+1}$. Then G contains at least 3-edge disjoint spanning trees.*

We constructed the extremal configurations below, which shows the bounds are best possible. To find a similar sufficient eigenvalue condition for more than 3 edge-disjoint spanning trees remains open due to the large increase in cases to consider and from using other results that are only known for small values. We conjecture the following.

Conjecture 5. *Let $d \ge 8$ and $4 \le k \le \lfloor \frac{d}{2} \rfloor$ be two integers. If G is a d-regular graph such that $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\sigma(G) \ge k$.*

Case of Theorem 3

The proof of both thereoems use the contrapositive. In Theorem 3, we assume a graph has exactly 1 edge disjoint spanning tree. Then by Theorem 2, there exists a partition of the vertex set with *less than* $2(t-1)$ edges going amongst the parts.

The restriction on the edges narrows down the possible structures of the graph. A possible example is

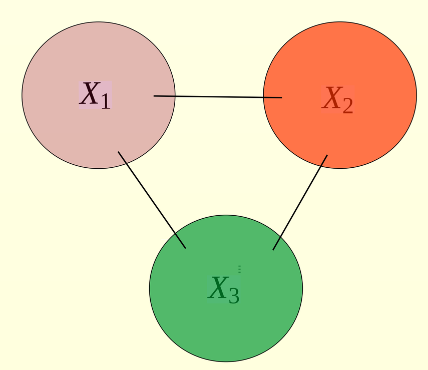


Figure 1: A possible structure of a graph with its vertex set partitioned into 3 parts.

The natural partition in 3 parts for a quotient matrix P, with a simple eigenvalue interlacing argument on $\lambda_2(P)$ and $\lambda_2(G)$ gives $\lambda_2(G) \ge d - \frac{3}{d+1}$ (since we are showing the contrapositive).

For larger t, calculating the eigenvalues becomes more difficult. Two cases of Theorem 4 are shown to the right, where a partition into $t = 4$ and 5 parts are considered.

Cases of Theorem 4

In Theorem 4, the contrapositive assumes there exists a partition of the vertex set with *less than* $3(t-1)$ edges going amongst the parts. Two possible structures of a graph are given below.

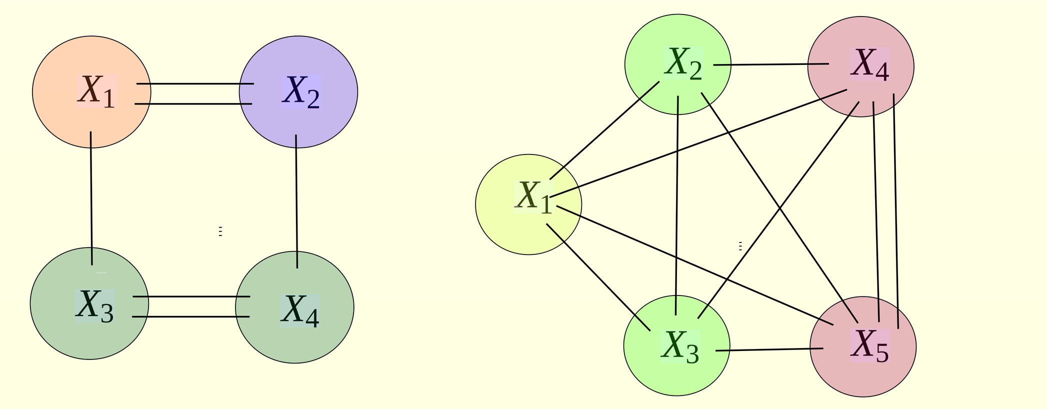


Figure 2: Two graphs with its vertex set partitioned into 4 and 5 parts.

Instead of considering the natural partition in 4 or 5 parts to represent a quotient matrix, the color coding shows we partition further into just 3 parts. Looking at its quotient matrix, we following another interlacing argument to conclude $\lambda_2 \ge d - \frac{5}{d+1}$.

For larger t, the restriction on edges amongst parts concludes there will be at least 2 parts with no edges between them. Analyzing the induced subgraphs on these parts directly, with eigenvalue interlacing, concludes $\lambda_2 \ge d - \frac{5}{d+1}$.

References

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