Bifurcations of a Prescribed Mean Curvature Equation

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Introduction

Mathematical models of the form $Hu = f(u)$, where $H$ is the mean curvature operator, are crucial in understanding capillary surfaces. They are particularly interesting mathematically due to the fact that many of their solutions set undergo intriguing bifurcations [see e.g., [PX11]]. Here, we study the solution set of the problem

$$\text{div} \frac{\nabla u}{\sqrt{1+\varepsilon^2|\nabla u|^2}} = \frac{\lambda}{1+u^2}, \quad x \in \Omega; \quad u = 0, \quad x \in \partial \Omega, \quad u(0) = u(1), \quad (1)$$

which derives from electrostatically deflecting a planar soap film. Specifically, we look at the two cases where $\Omega = [-1,1]$ and $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$. Note that here $\lambda$, which characterizes the applied voltage, and $\varepsilon$, which characterizes the size the undecayed planar soap film, are nonnegative, dimensionless parameters.

One-dimensional

The 1D version of (1) reduces via symmetry to

$$\frac{u''}{\sqrt{1+\varepsilon^2|u'|^2}} = \frac{\lambda}{1+u^2}, \quad 0 < x < 1; \quad u'(0) = u(1) = 0, \quad (2)$$

which has the first integral $\varepsilon^2(1+\varepsilon^2|u'|^2)^{-1/2} - \lambda(1+u)^{-1} = E$. Therefore, in solving for $u'$ and separating variables yields the following.

**Lemma.** The values $(\lambda, \varepsilon, \alpha)$ give a solution $u$ of the ordinary differential equation (2), with $u(0) = \alpha$, if and only if $T(\alpha; \lambda, \varepsilon) = 1$, where $E = \varepsilon^2 - \lambda/(1+\alpha)$ and

$$T(\alpha; \lambda, \varepsilon) := \int_0^\alpha \frac{\varepsilon^2[(1+E)(1+z)]}{(1+z)^2 - \varepsilon^2[(1+E)(1+z)]} \, dz. \quad (3)$$

From this we have

**Theorem 1.** There exists an $\varepsilon^* > 0$ such that

(i) if $\varepsilon < \varepsilon^*$, then there exists a value $\lambda^*(\varepsilon)$ such that (a) for $\lambda \in (0, \lambda^*)$, (2) has exactly two solutions; (b) for $\lambda = \lambda^*$, (2) has exactly one solution; (c) for $\lambda > \lambda^*$, (2) has no solutions.

(ii) if $\varepsilon > \varepsilon^*$, then there exists three values $\lambda_0$, $\lambda_1$, and $\lambda_2$, which depend on $\varepsilon$, such that (a) for $\lambda \in (0, \lambda_0] \cup [\lambda_2, \lambda^*)$, (2) has exactly two solutions; (b) for $\lambda \in (\lambda_1, \lambda_2) \cup \{\lambda^*\}$, (2) has exactly one solution; (c) for $\lambda > \lambda^*$, (2) has no solutions.

Result: The solutions set of (2) undergoes a **splitting bifurcation** at $\varepsilon = \varepsilon^* \approx 2.857$, i.e., when $\varepsilon$ transitions from less than or equal to to greater than $\varepsilon^*$, the upper solution branch splits into two parts (see the middle and bottom subfigures of Fig. 1).

Two-dimensional

The 2D version of (1) with $\Omega$ equal to the unit disk whose far field behavior is reduces to

$$1 \cdot \frac{r u'}{\sqrt{1+r^2|u'|^2}} = \frac{\lambda}{1+u^2}, \quad 0 < r < 1; \quad u'(0) = u(1) = 0. \quad (3)$$

This problem exhibits a **dead-end bifurcation** (see Figure 2).

- If $\varepsilon = 0$, then for all $\alpha \in (-1,0]$ there exists a solution $u$ of (3) such that $u(0) = \alpha$ [PB03].
- However, for $\varepsilon > 0$, there exists an $\alpha_0(\varepsilon) \in (-1, 0)$ such that $u$ is a solution of (3), then $u(0) < \alpha_0$.

**Figure 2.** Left: Bifurcation curves of (3) computed for $\varepsilon = 0.05, 0.1, 0.5, 1, 2$ (from right to left). Note that at this scale the $\varepsilon = 0.05$ and $\varepsilon = 0.1$ curves appear equal.

**Right:** Magnified portion of the left fig. Here, $\varepsilon = 0.05, 0.1, 0.5$ curves are seen. Note that all of the curves stop before $u(0)$ reaches 1.

**Asymptotic analysis.** To analyze the dead-end bifurcation for $\varepsilon \ll 1$, we look at (3) with the point constraint $u(0) = 1+\delta$, for $0 < \delta \ll 1$. Since the problem involves two small parameters, the analysis must be performed in the distinguished limit $\varepsilon^2/\delta = 0$, where $\delta = O(1)$. Expanding $u$ and $\lambda$ as $u \sim u_0 + \varepsilon^2 u_1$, and $\lambda \sim \lambda_0 + \varepsilon^2 \lambda_1$, leads to a singular perturbation problem with a boundary layer of width $O(\delta^{3/2})$ at $r = 0$. The leading order inner problem is

$$1 \cdot \frac{\rho u''}{\sqrt{1+\rho^2(\rho u_0')^2}} = \frac{\lambda_0}{1+u_0^2}, \quad 0 < \rho < \infty; \quad u_0(0) = 1, \quad u_0'(0) = 0. \quad (4)$$

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**References**
