Erdős-Ko-Rado Theorems for Permutations

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(joint work with Bahman Ahmadi, Chris Godsil, Alison Purdy, Pablo Spiga and Pham Huu Tiep)

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Erdős-Ko-Rado Theorem

Theorem (EKR Theorem-1961)

Let $\mathcal{F}$ be an intersecting $k$-set system on an $n$-set. If $n > 2k$, then

1. $|\mathcal{F}| \leq \binom{n-1}{k-1},$

2. and $\mathcal{F}$ meets this bound if and only if the sets all contain a common element.

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This is the largest intersecting 3-set system from $[1..8]$.

123, 124, 125, 126, 127, 128, 134,
135, 136, 137, 138, 145, 146, 147,
148, 156, 157, 158, 167, 168, 178
Reasons to love the Erdős-Ko-Rado Theorem

There are many different proofs.
  - Original used compression and counting,
  - Katona’s cycle proof is very accessible
  - Algebraic eigenvalue proof

There are many extensions of this theorem.
  - What is the largest intersecting system without a common point?
  - What is largest $t$-intersecting system?
  - What is the largest cross-intersecting system?

The EKR theorem generalizes to many different objects.
  - $k$-dimensional vector subspaces over a finite field,
  - Length-$n$ sequences in $\mathbb{Z}_q$,
  - Integer partitions,
  - Domino tilings,
  - Permutations . . .

Which proofs can be extend to other objects?
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Which proofs can be extend to other objects?
The Kneser graph $K(n, k)$ is the graph with all $k$-subsets of an $n$-set as the vertices and vertices are adjacent if they are not intersecting.
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**Figure:** The Kneser Graph $K(5, 2)$, or our old friend Petersen.
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Figure: The Kneser Graph $K(5, 2)$, or our old friend Petersen.

What is the largest **coclique/independent set** in this graph?
Kneser Graphs

Properties of $K(n, k)$

- Vertex transitive and regular, with degree $\binom{n-k}{k}$.
- It is a graph in the Johnson association scheme.
- If $n \geq 2k$, eigenvalues are

$$(-1)^i \binom{n-k-i}{k-i} \quad \text{with multiplicity} \quad \binom{n}{i} - \binom{n}{i-1}.$$
Delsarte-Hoffman Ratio Bound for Cocliques

**Theorem (bound part)**

If $X$ is a $d$-regular graph, then

$$
\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}
$$

where $\tau$ is the least eigenvalue of $A(X)$ (or a weighted adjacency matrix).
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Theorem (characterization part)

If equality holds in the ratio bound and \( v_S \) is a characteristic vector for a maximum coclique \( S \), then

\[
v_S - \frac{\alpha(X)}{|V(X)|} 1
\]

is an eigenvector for \( \tau \).
Ratio Bound for Kneser Graph

Bound on the size of a coclique:

\[ \alpha(K(n, k)) \leq \frac{\binom{n}{k}}{1 - \frac{\binom{n-k}{k}}{\binom{n-k-1}{k-1}}} = \frac{n}{k-1}. \]
Bound on the size of a coclique:

\[ \alpha(K(n, k)) \leq \frac{\binom{n}{k}}{1 - \frac{\binom{n-k}{k}}{-\binom{n-k-1}{k-1}}} = \binom{n-1}{k-1}. \]

Characterization:

- 1 is a \( \binom{n-k}{k} \)-eigenvector.
- Let \( v_i \) be the characteristic vector of the collection of all sets that contain \( i \). The vectors \( v_i - \frac{k}{n}1 \) are \( -\binom{n-k-1}{k-1} \)-eigenvectors.
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Characterization:

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- Let $v_i$ be the characteristic vector of the collection of all sets that contain $i$. The vectors $v_i - \frac{k}{n} 1$ are $-\binom{n-k-1}{k-1}$-eigenvectors.
- $v_i$ span the $\binom{n-k}{k}$-eigenspace and the $-\binom{n-k-1}{k-1}$-eigenspace.
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- \( v_i \) span the \( \binom{n-k}{k} \)-eigenspace and the \( -\binom{n-k-1}{k-1} \)-eigenspace.
- The characteristic vector for any maximum coclique is a linear combination of the \( v_i \).
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Characterization:

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- Let \( v_i \) be the characteristic vector of the collection of all sets that contain \( i \). The vectors \( v_i - \frac{k}{n} 1 \) are \( -\binom{n-k-1}{k-1} \)-eigenvectors.
- \( v_i \) span the \( \binom{n-k}{k} \)-eigenspace and the \( -\binom{n-k-1}{k-1} \)-eigenspace.
- The characteristic vector for any maximum coclique is a linear combination of the \( v_i \).
- If \( n > 2k \), the only linear combinations that give 01-vector with weight \( \binom{n-1}{k-1} \) is \( v_i \).
Extending this Method to Other Objects

1. For objects made of “atoms”, two objects are intersecting if they have a common atom.
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The canonically intersecting set are maximum intersecting sets.
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The characteristic vector of any max intersecting set a linear combination of the canonically intersecting sets.
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**The strict EKR property**

All the maximum intersecting sets the canonically intersecting sets?
Steps in this Type of Proof

1. Define a derangement graph:
   - The objects are the vertices of the graph, and
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4. Find all linear combinations of characteristic vectors of the canonically intersecting objects that give characteristic vectors on intersecting sets.
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This method works for lots of objects—we will consider permutations.
Two permutations $\sigma, \pi \in \text{Sym}(n)$ intersect if for some $i \in [1..n]$. 

$$\sigma(i) = \pi(i) \quad \text{or} \quad \pi^{-1}\sigma(i) = i.$$
Intersecting Permutations

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A permutation is a derangement if it fixes no points. So \( \sigma \) and \( \pi \) are intersecting if and only if \( \pi^{-1}\sigma \) is not a derangement.
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A permutation is a derangement if it fixes no points. So $\sigma$ and $\pi$ are intersecting if and only if $\pi^{-1}\sigma$ is not a derangement.

Define $S_{i,j}$ to be the set of all permutations in $G$ that map $i$ to $j$.

1. These are the cosets of the stabilizers of a point.
2. These are the canonical cocliques in $\Gamma_G$.
3. Use $v_{i,j}$ to denote the characteristic vector of $S_{i,j}$. 
For any $G \leq \text{Sym}(n)$ we can define a *Derangement Graph*.

- $\Gamma_G$ denotes the derangement graph for a group $G$.
- The vertices are the elements of $G$.
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- $\Gamma_G = \text{Cay}(G, \text{Der}(G))$ where $\text{Der}(G)$ is the set of derangements of $G$. 


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This derangement graph depends on the action!
Examples Derangement Graph

Figure: The graph $\Gamma_{D(4)}$. 
If $G$ is cyclic, then $\Gamma_G$ is the complement of the circulant graph $(\mathbb{Z}_{|G|}, C)$ where $C$ is the set of all multiples of the cycle lengths of the generator.

Figure: The graph $G = \langle (1, 2, 3)(4, 5) \rangle$. 
Derangement Graph for $\text{Sym}(4)$
Properties of the Derangement Graph

- $\Gamma_G$ is vertex transitive.
- An intersecting set in $G$ is a coclique in $\Gamma_G$.
- If $G$ has a sharply 1-transitive set, is a clique of size $n$. 
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$$\frac{1}{\phi(1)} \sum_{d \in D} \phi(d)$$

where $\phi$ is an irreducible character of $G$. 
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  where $\phi$ is an irreducible character of $G$.
- The trivial representation gives the degree, $|D|$.
- The eigenspaces are unions of $G$-modules, projections are given by $E_{\phi}$ (matrix with $(g, h)$-entry $\phi(g^{-1}h)$).
- It is the union of the graphs in the conjugacy class association scheme (the $E_{\phi}$ are the idempotents in the scheme).
2-Transitive Subgroups

1. The permutation character is $\text{fix}(g)$.
2. Define $\chi(g) = \text{fix}(g) - 1$ ($\chi = \text{permutation} - \text{trivial}$).
2-Transitive Subgroups

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2. Define $\chi(g) = \text{fix}(g) - 1$ ($\chi$ = permutation – trivial).
3. If $G$ is 2-transitive, $\chi$ is irreducible.

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} (|\text{fix}(g)| - 1)^2$$

$$= \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|^2 - 2 \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)| + \frac{1}{|G|} \sum_{g \in G} 1 = 1$$
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4. The eigenvalue for $\chi$ is

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{g \in D} \chi(g) = -\frac{|D|}{n - 1}$$
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The eigenvalue for $\chi$ is

\[
\lambda_{\chi} = \frac{1}{\chi(1)} \sum_{g \in D} \chi(g) = \frac{-|D|}{n - 1}
\]

The permutation module is the span of the columns of $E_{\chi}$ and $E_1$ (the all ones vector).
Apply the Ratio Bound

**Theorem**

Let $G$ be a 2-transitive group acting on an $n$-set. If $\frac{-|D|}{n-1}$ is the least eigenvalue for $\Gamma_G$, then the largest intersecting set has size $\frac{|G|}{n}$.
Apply the Ratio Bound

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**Proof.** By the ratio bound

$$\alpha(\Gamma_G) \leq \frac{|G|}{1 - \frac{|D|}{n-1}} = \frac{|G|}{n}.$$ 

Since $G$ is transitive, then the stabilizer of a point has size $|G|/n$. 

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**Theorem**

Further, if only $\chi$ has eigenvalue $\frac{-|D|}{n-1}$, then the characteristic vector of any maximum coclique $S$ is in the permutation module.

**Proof.** The $\chi$-module is the entire $\frac{-|D|}{n-1}$-eigenspace, and $v_S - \frac{1}{n} 1$ is a $\frac{-|D|}{n-1}$-eigenvector.
A Basis for the Permutation Module

**Lemma**

If $G$ is a 2-transitive group, then $v_{i,j}$ is in the permutation module.

**Proof.**

$$E_X(v_{i,j} - \frac{1}{n}1) = E_X(v_{i,j}) = v_{i,j} - \frac{1}{n}1.$$
Lemma

If $G$ is a 2-transitive group, then $v_{i,j}$ is in the permutation module.

Proof.

$$E_{\chi}(v_{i,j} - \frac{1}{n}1) = E_{\chi}(v_{i,j}) = v_{i,j} - \frac{1}{n}1.$$ 

Lemma

Let $B := \{v_{i,j} - \frac{1}{n}1 | i, j \in [n - 1]\}$

is a basis for the $\chi$-module of $G$. 
A Basis for the Permutation Module

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Proof. Define a matrix \( L \) with columns \( v_{i,j} \).

\[
L^\top L = \frac{(n-1)!}{2} I_{(n-1)^2} + \frac{(n-2)!}{2} (A(K_{n-1}) \otimes A(K_{n-1})).
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Corollary

\[
\{v_{i,j} | i, j \in [n-1]\} \cup v_{n,n} \text{ is a basis for the permutation module of } G.
\]
The Characterization for \( \text{Sym}(3) \)

If the characteristic vector for any maximum coclique is in the permutation module, then it is a linear combination of the vectors \( v_{i,j} \).
The Characterization for $\text{Sym}(3)$

If the characteristic vector for any maximum coclique is in the permutation module, then it is a linear combination of the vectors $v_{i,j}$.

For $\text{Sym}(3)$,

<table>
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<th>2→2</th>
<th>3→3</th>
<th>1→2</th>
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\[
\begin{pmatrix}
  x_{1,1} \\
  x_{2,2} \\
  x_{3,3} \\
  x_{1,2} \\
  x_{2,1}
\end{pmatrix}
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If the characteristic vector for any maximum coclique is in the permutation module, then it is a linear combination of the vectors $v_{i,j}$.

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\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
x_{1,1} \\
x_{2,2} \\
x_{3,3} \\
x_{1,2} \\
x_{2,1}
\end{pmatrix}
= \begin{pmatrix}
1_{[1,2,3]} \\
0_{[2,3,1]} \\
0_{[3,1,2]} \\
y_{[1,3,2]} \\
y_{[3,2,1]} \\
y_{[2,1,3]}
\end{pmatrix}
\]
Characterization for General 2-Transitive Groups

\[
\begin{pmatrix}
1 & 0 \\
0 & N \\
X & Y
\end{pmatrix}
\]

steps to characterize maximum cocliques

1. If the matrix \( N \) has full rank then \( x_2 = 0 \). (HARD!!)

2. The matrix \( X \) contains a \( n \times n \) identity matrix, so \( x_1 \) is a 01-vector.

3. The sum of the entries of \( x_1 \) is 1, so \( x_1 \) contains exactly one 1.

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1. If the group doesn't have the strict EKR property, we may get a characterization by finding a basis of the kernel of \( N \).

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Steps to characterize maximum cocliques

\[
\begin{pmatrix}
1 & 0 \\
0 & N \\
x_1 & x_2
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
y'
\end{pmatrix}
\]
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i \rightarrow i || i \rightarrow j
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1. If the group doesn’t have the strict EKR property, we may get a characterization by finding a basis of the kernel of $N$.
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All 2-transitive groups have EKR-property (M., Spiga, Tiep).

Sym\( (n) \) has strict EKR-property.

For PGL\( (n, q) \)

- all \( n \) it has the EKR-module property (Spiga);
- for \( n = 2 \) has the strict-EKR property (M. and Spiga);
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Long, Plaza, Sin, Xiang showed that PSL\( (2, q) \) has the strict-EKR property.

Ahmadi and M. showed Alt\( (n) \) and the Matthieu groups have the strict EKR.

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Results for 2-Transitive Groups

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6. PSU$(3, q)$ has the EKR-module property.

I don’t know of an example of 2-transitive group that does not have the EKR-module property.
Let $G$ be a 1-transitive group.

1. Canonical cocliques are still the sets $S_{i,j}$ of all permutations that map $i$ to $j$.

2. Since $G$ is 1-transitive, $|S_{i,j}| = \frac{|G|}{n}$.

3. $\chi(g) = \text{fix}(g) - 1$ is still a representation, just not irreducible!
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Can we prove that the characteristic vector of a maximum coclique is in the permutation module?
Clique-Coclique Bound in an Association Scheme

Set up:
- If $X$ is a graph in an association scheme.

\[ \alpha(X) \omega(X) \leq |V(X)|. \]
Clique-Coclique Bound in an Association Scheme

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More Set up:
- Equality in the clique-Coclique bound.
- Let $\mathcal{E} = \{E_0, E_1, \ldots, E_d\}$ the idempotents of the scheme,
- If $C$ is a clique of maximum size with characteristic vector $v_C$ and
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$v_S^T E_i v_S \ v_C^T E_i v_C = 0,$
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\[ v_S^T E_i v_S v_C^T E_i v_C = 0, \]

So, if $E_i v_C \neq 0$, then $E_i v_s = 0.$
The group $\text{GL}(2, q)$ acts on the $q^2 - 1$ non-zero vectors in $\mathbb{F}_q^2$.

1. This action is 1-transitive (not 2-transitive).
2. This group has a clique of size $q^2 - 1$.
3. The subgroup that fixes a point is a subgroup of size $q(q - 1)$,
4. The eigenvalues of the derangement graph are:

$$q(q^3 - 2q^2 - q + 3), \quad q, \quad -q^2 + 2q, \quad -q^2 + q + 1.$$

The ratio bound does not hold with equality, but it has EKR property by clique-coclique bound.
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I can put a weighting on the conjugacy classes and then calculate the eigenvalues of the weighted adjacency matrix.

<table>
<thead>
<tr>
<th>Character degree number</th>
<th>Unweighted evalue</th>
<th>Weighted evalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = \beta q^2 - 3$</td>
<td>$q^{2-2}$ or $q^{2-4}$</td>
<td>$q^{2} - 3$ or $q^{2} - 2$</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>$q^{2-2}$ or $q^{2-4}$</td>
<td>$q^{2} - 3$ or $q^{2} - 2$</td>
</tr>
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<td>$q^{2-2}$ or $q^{2-4}$</td>
<td>$q^{2} - 3$ or $q^{2} - 2$</td>
</tr>
<tr>
<td>$\chi = 1$</td>
<td>$q^{2-2}$ or $q^{2-4}$</td>
<td>$q^{2} - 3$ or $q^{2} - 2$</td>
</tr>
<tr>
<td>$\chi \neq 1$</td>
<td>$q^{2-2}$ or $q^{2-4}$</td>
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</tr>
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<tbody>
<tr>
<td>$\alpha = 1$</td>
<td>$q + 1$</td>
<td>$\frac{q-3}{2}$ or $\frac{q-2}{2}$</td>
<td>$q$</td>
<td>$-1$</td>
</tr>
<tr>
<td>otherwise</td>
<td>$q + 1$</td>
<td>$q - 2$</td>
<td>$-q^2 + 2q$</td>
<td>$2\frac{1}{q-3}$</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>$q$</td>
<td>$1$</td>
<td>$-q^2 + q + 1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\alpha^2 = 1$</td>
<td>$q$</td>
<td>$1$</td>
<td>$q$</td>
<td>$-1$</td>
</tr>
<tr>
<td>otherwise</td>
<td>$q$</td>
<td>$q - 3$</td>
<td>$q$</td>
<td>$\frac{1}{q} \left( \frac{q-1}{q-2} + \frac{q+1}{q-3} \right)$</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$q(q^3 - 2q^2 - q + 3)$</td>
<td>$q^2 - 2$</td>
</tr>
<tr>
<td>$\alpha^2 = 1$ (if $q$ is odd)</td>
<td>$1$</td>
<td>$1$</td>
<td>$q$</td>
<td>$-1$</td>
</tr>
<tr>
<td>otherwise</td>
<td>$1$</td>
<td>$q - 3$</td>
<td>$q$</td>
<td>$\frac{q-1}{q-2} + \frac{q+1}{q-3}$</td>
</tr>
<tr>
<td>$\chi = 1$</td>
<td>$q - 1$</td>
<td>$\frac{q-1}{2}$ or $\frac{q}{2}$</td>
<td>$q$</td>
<td>$q - 3$</td>
</tr>
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Big Take Aways:

1. The trivial representation gives the maximum eigenvalue $q^2 - 2$.
2. All the other irreducible representations in the permutation representation all give the minimal eigenvalue $-1$ (and some other representations too).
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Can we do this for any 1-transitive group?
Consider $\text{Sym}(n)$ acting on ordered $t$-sets.

**Theorem (Ellis, Friedgut, Pilpel 2010)**

For $n$ sufficiently large, the maximum $t$-intersecting set of permutations has size $(n - t)!$ and is the coset of the point-wise stabilizer of a $t$-set.
$t$-Intersecting Permutations

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AGL(2, q) on Lines

AGL(2, q) acts on the $q(q + 1)$ lines of $\mathbb{F}_q^2$, this action is 1-transitive.

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Conjecture

The group AGL(2, q) does not have the EKR property. There is an intersecting set of size \( \frac{1}{2} q^2 (q - 1)(3q - 4) \).

The stabilizer of a point has size \( \frac{q^3(q-1)^2(q+1)}{q(q+1)} = q^2(q - 1)^2 \).
Big Dramatic Summary

2-Transitive Groups

1. All have the EKR property.
2. I conjecture that all have the EKR module property.
3. Don’t all have strict EKR property.
   a. Can we characterize which do?
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2-Transitive Groups

1. Do not all have the EKR property.
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What other families of groups are interesting?
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