Erdős-Ko-Rado Theorems for Permutations

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Erdős-Ko-Rado Theorem

Theorem (EKR Theorem-1961)

Let $\mathcal{F}$ be an intersecting k-set system on an n-set. If $n > 2k$, then

1. $|\mathcal{F}| \leq \binom{n-1}{k-1}$,
2. *and $\mathcal{F}$ meets this bound if and only if the sets all contain a common element.*

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This is the largest intersecting 3-set system from $[1..8]$.

123,	124,	125,	126,	127,	128,	134,
135,	136,	137,	138,	145,	146,	147,
148,	156,	157,	158,	167,	168,	178

Reasons to love the Erdős-Ko-Rado Theorem

- There are many different proofs.
  - Original used compression and counting,
  - Katona's cycle proof is very accessible
  - Algebraic eigenvalue proof

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- The EKR theorem generalizes to many different objects.
  - k-dimensional vector subspaces over a finite field,
  - Length-n sequences in $\mathbb{Z}_q$,
  - Integer partitions,
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Which proofs can be extend to other objects?

Algebraic Graph Theory Proof

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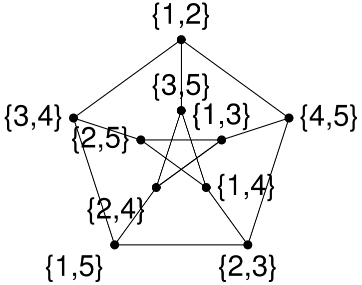

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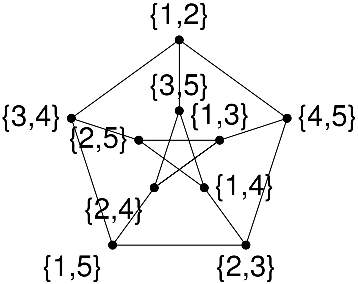

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What is the largest **coclique/independent set** in this graph?

Kneser Graphs

Properties of $K(n,k)$

- Vertex transitive and regular, with degree $\binom{n-k}{k}$.
- It is a graph in the Johnson association scheme.
- If $n \geq 2k$, eigenvalues are

$$(-1)^i \binom{n-k-i}{k-i} \quad \text{with multiplicity } \binom{n}{i} - \binom{n}{i-1}.$$

Delsarte-Hoffman Ratio Bound for Cocliques

Theorem (bound part)

If X is a d-regular graph, then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}$$

where τ is the least eigenvalue of $A(X)$ (or a weighted adjacency matrix).

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Theorem (characterization part)

If equality holds in the ratio bound and v_S is a characteristic vector for a maximum coclique S, then

$$v_S - \frac{\alpha(X)}{|V(X)|}\mathbf{1}$$

is an eigenvector for τ.

Ratio Bound for Kneser Graph

Bound on the size of a coclique:

$$\alpha(K(n,k)) \leq \frac{\binom{n}{k}}{1 - \frac{\binom{n-k}{k}}{-\binom{n-k-1}{k-1}}} = \binom{n-1}{k-1}.$$

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Characterization:

- $\mathbf{1}$ is a $\binom{n-k}{k}$-eigenvector.
- Let v_i be the characteristic vector of the collection of all sets that contain i. The vectors $v_i - \frac{k}{n}\mathbf{1}$ are $-\binom{n-k-1}{k-1}$-eigenvectors.

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- v_i span the $\binom{n-k}{k}$-eigenspace and the $-\binom{n-k-1}{k-1}$-eigenspace.
- The characteristic vector for any maximum coclique is a linear combination of the v_i.
- If $n > 2k$, the only linear combinations that give 01-vector with weight $\binom{n-1}{k-1}$ is v_i.

Extending this Method to Other Objects

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Steps in this Type of Proof

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This method works for lots of objects–we will consider permutations.

Intersecting Permutations

Two permutations $\sigma, \pi \in \mathrm{Sym}(n)$ *intersect* if for some $i \in [1..n]$.

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Define $S_{i,j}$ to be the set of all permutations in G that map i to j.

1. These are the cosets of the stabilizers of a point
2. These are the *canonical* cocliques in Γ_G.
3. Use $v_{i,j}$ to denote the characteristic vector of $S_{i,j}$.

Derangement Graph

For any $G \leq \mathrm{Sym}(n)$ we can define a *Derangement Graph*.

- Γ_G denotes the derangement graph for a group G.
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This derangement graph depends on the action!

Examples Derangement Graph

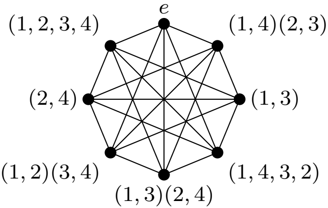


Figure: The graph $\Gamma_{D(4)}$.

Derangement Graph for Cyclic Groups

If G is cyclic, then Γ_G is the complement of the circulant graph $(\mathbb{Z}_{|G|}, C)$ where C is the set of all multiples of the cycle lengths of the generator.

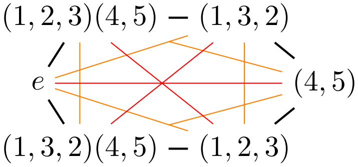
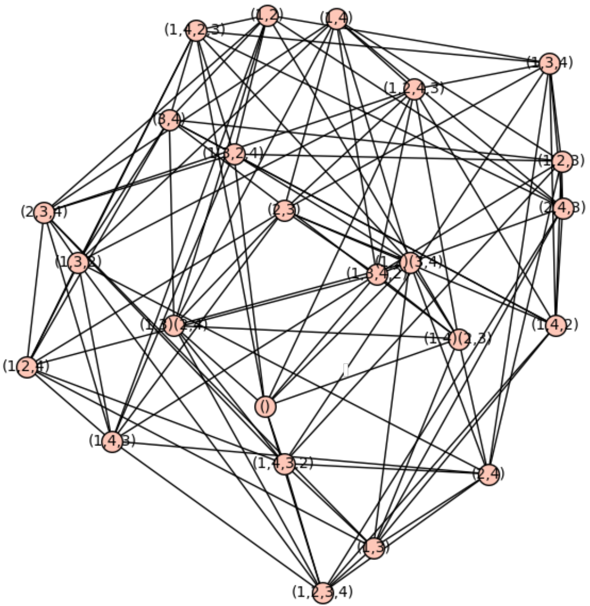


Figure: The graph $G = \langle (1,2,3)(4,5) \rangle$.

Derangement Graph for $\mathrm{Sym}(4)$



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- The eigenspaces are unions of G-modules, projections are given by E_ϕ (matrix with (g, h)-entry $\phi(g^{-1}h)$).
- It is the union of the graphs in the conjugacy class association scheme (the E_ϕ are the idempotents in the scheme).

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5. The *permutation module* is the span of the columns of E_χ and $E_{\mathbf{1}}$ (the all ones vector).

Apply the Ratio Bound

Theorem

Let G be a 2-transitive group acting on an n-set. If $\frac{-|D|}{n-1}$ is the least eigenvalue for Γ_G, then the largest intersecting set has size $\frac{|G|}{n}$

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Proof. By the ratio bound

$$\alpha(\Gamma_G) \leq \frac{|G|}{1 - \frac{|D|}{-\frac{|D|}{n-1}}} = \frac{|G|}{n}.$$

Since G is transitive, then the stabilizer of a point has size $|G|/n$.

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Theorem

Further, if only χ has eigenvalue $\frac{-|D|}{n-1}$, then the characteristic vector of any maximum coclique S is in the permutation module.

Proof. The χ-module is the entire $\frac{-|D|}{n-1}$-eigenspace, and $v_S - \frac{1}{n}\mathbf{1}$ is a $\frac{-|D|}{n-1}$-eigenvector.

A Basis for the Permutation Module

Lemma

G is a 2-transitive group, then $v_{i,j}$ is in the permutation module.

Proof.

$$E_\chi(v_{i,j} - \frac{1}{n}\mathbf{1}) = E_\chi(v_{i,j}) = v_{i,j} - \frac{1}{n}\mathbf{1}.$$

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Proof. Define a matrix L with columns $v_{i,j}$.

$$L^\top L = \frac{(n-1)!}{2} I_{(n-1)^2} + \frac{(n-2)!}{2} \left(A(K_{n-1}) \otimes A(K_{n-1})\right).$$

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Corollary

$\{v_{i,j} \mid i, j \in [n-1]\} \cup v_{n,n}$ is a basis for the permutation module of G.

The Characterization for $\mathrm{Sym}(3)$

If the characteristic vector for any maximum coclique is in the permutation module, then it is a linear combination of the vectors $v_{i,j}$.

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	1→1	2→2	3→3	1→2	2→1
[1,2,3]	1	1	1	0	0
[2,3,1]	0	0	0	1	0
[3,1,2]	0	0	0	0	1
[1,3,2]	1	0	0	0	0
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[2,1,3]	0	0	1	1	1

$$\begin{pmatrix} x_{1,1} \\ x_{2,2} \\ x_{3,3} \\ x_{1,2} \\ x_{2,1} \end{pmatrix} = \begin{pmatrix} 1_{[1,2,3]} \\ 0_{[2,3,1]} \\ 0_{[3,1,2]} \\ y_{[1,3,2]} \\ y_{[3,2,1]} \\ y_{[2,1,3]} \end{pmatrix}$$

Characterization for General 2-Transitive Groups

	i → i	i → j
identity	**1**	0
derangements	0	N
other permutations	X	Y

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ y' \end{pmatrix}$$

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Steps to characterize maximum cocliques

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Some Notes

1. If the group doesn't have the strict EKR property, we may get a characterization by finding a basis of the kernel of N.
2. The 01-vectors y can be thought of as Cameron-Leibler sets.

Results for 2-Transitive Groups

1. All 2-transitive groups have EKR-property (M., Spiga, Tiep).
2. $\mathrm{Sym}(n)$ has strict EKR-property.
3. For $\mathrm{PGL}(n, q)$
   - all n it has the EKR-module property (Spiga);
   - for $n = 2$ has the strict-EKR property (M. and Spiga);
   - for $n \geq 3$ the maximum intersecting sets are either stabilizers of a point or a hyperplane (M. and Spiga, Spiga).
4. Long, Plaza, Sin, Xiang showed that $\mathrm{PSL}(2, q)$ has the strict-EKR property.
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I don't know of an example of 2-transitive group that does not have the EKR-module property.

1-Transitive Groups

Let G be a 1-transitive group.

1. Canonical cocliques are still the sets $S_{i,j}$ of all permutations that map i to j
2. Since G is 1-transitive, $|S_{i,j}| = \frac{|G|}{n}$.
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Can we prove that the characteristic vector of a maximum coclique is in the permutation module?

Clique-Coclique Bound in an Association Scheme

Set up:

- If X is a graph in an association scheme.

$$\alpha(X)\,\omega(X) \leq |V(X)|.$$

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- Equality in the clique-Coclique bound.
- Let $\mathcal{E} = \{E_0, E_1, \ldots, E_d\}$ the idempotents of the scheme,
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So, if $E_i v_C \neq 0$, then $E_i v_s = 0$.

General Linear Group $\mathrm{GL}(2, q)$

The group $\mathrm{GL}(2, q)$ acts on the $q^2 - 1$ non-zero vectors in $\mathbb{F}_q^2$.

1. This action is 1-transitive (not 2-transitive).
2. This group has a clique of size $q^2 - 1$.
3. The subgroup that fixes a point is a subgroup of size $q(q-1)$,
4. The eigenvalues of the derangement graph are:

$$q(q^3 - 2q^2 - q + 3), \quad q, \quad -q^2 + 2q, \quad -q^2 + q + 1.$$

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The ratio bound does not hold with equality, but it has EKR property by clique-coclique bound.

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Character	degree	number	unweighted evalue	weighted evalue
$\alpha = \overline{\beta}$	$q+1$	$\frac{q-3}{2}$ or $\frac{q-2}{2}$	q	-1
$\alpha = 1$	$q+1$	$q-2$	$-q^2+2q$	-1
otherwise	$q+1$	$\frac{(q-3)^2}{2}$ or $\frac{(q-2)(q-4)}{2}$	q	$\frac{2}{q-3}$
$\alpha = 1$	q	1	$-q^2+q+1$	-1
$\alpha^2 = 1$	q	1	q	-1
otherwise	q	$q-3$	q	$\frac{1}{q}\left(\frac{q-1}{q-2}+\frac{q+1}{q-3}\right)$
$\alpha = 1$	1	1	$q(q^3-2q^2-q+3)$	q^2-2
$\alpha^2 = 1$ (if q is odd)	1	1	q	-1
otherwise	1	$q-3$	q	$\frac{q-1}{q-2}+\frac{q+1}{q-3}$
$\chi = 1$	$q-1$	$\frac{q-1}{2}$ or $\frac{q}{2}$	q	$q-3$
$\chi \neq 1$	$q-1$	$\frac{(q-1)^2}{2}$ or $\frac{q(q-2)}{2}$	q	$\frac{2}{q-2}$

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Can we do this for any 1-transitive group?

t-Intersecting Permutations

Consider $\mathrm{Sym}(n)$ acting on ordered t-sets.

Theorem (Ellis, Friedgut, Pilpel 2010)

For n sufficiently large, the maximum t-intersecting set of permutations has size $(n-t)!$ and is the coset of the point-wise stabilizer of a t-set.

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Consider $\mathrm{Sym}(n)$ acting on unordered t-sets.

Theorem (Ellis, 2011)

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$\mathrm{AGL}(2,q)$ on Lines

$\mathrm{AGL}(2,q)$ acts on the $q(q+1)$ lines of $\mathbb{F}_q^2$, this action is 1-transitive.

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Conjecture

The group $\mathrm{AGL}(2, q)$ does not have the EKR property. There is an intersecting set of size $\frac{1}{2}q^2(q-1)(3q-4)$.

The stabilizer of a point has size $\frac{q^3(q-1)^2(q+1)}{q(q+1)} = q^2(q-1)^2$.

Big Dramatic Summary

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    a. Can we characterize which do?
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What other families of groups are interesting?