The underlying geometry of the graphs $CD(k, q)$

Andrew Woldar
Villanova University

Algebraic and Extremal Graph Theory Conference
University of Delaware, August 7-10, 2017

To Ron Solomon, who gave me the gift of knowledge and the rewards of friendship, *"Semper gratiam habebo."*

Introduction

The graphs $CD(k, q)$ exhibit many interesting extremal properties, and have sundry applications to coding theory, cryptography and network topology.

Introduction

The graphs $CD(k, q)$ exhibit many interesting extremal properties, and have sundry applications to coding theory, cryptography and network topology.

They have the greatest number of edges among all known graphs of a fixed order and girth at least g for $g \geq 5$, $g \neq 11, 12$. (For $g = 11, 12$, they are outperformed by the generalized hexagon of type G_2.)

Introduction

The graphs $CD(k, q)$ exhibit many interesting extremal properties, and have sundry applications to coding theory, cryptography and network topology.

They have the greatest number of edges among all known graphs of a fixed order and girth at least g for $g \geq 5$, $g \neq 11, 12$. (For $g = 11, 12$, they are outperformed by the generalized hexagon of type G_2.)

They are very close to being Ramanujan, with a conjectured bound of $\lambda_2 \leq 2\sqrt{q}$ where λ_2 is the second largest eigenvalue.

Introduction

The graphs $CD(k, q)$ exhibit many interesting extremal properties, and have sundry applications to coding theory, cryptography and network topology.

They have the greatest number of edges among all known graphs of a fixed order and girth at least g for $g \geq 5$, $g \neq 11, 12$. (For $g = 11, 12$, they are outperformed by the generalized hexagon of type G_2.)

They are very close to being Ramanujan, with a conjectured bound of $\lambda_2 \leq 2\sqrt{q}$ where λ_2 is the second largest eigenvalue.

$$\left(\text{Ramanujan} \iff \lambda_2 \leq 2\sqrt{q-1} \right)$$

Introduction

Graphs $CD(k, q)$ are derived from the graphs $D(k, q)$, which are examples of **algebraically defined graphs**, that is, graphs whose vertices are coordinate vectors, with two vertices adjacent provided their coordinates satisfy a prescribed set of equations (**adjacency relations**).

Introduction

Graphs $CD(k,q)$ are derived from the graphs $D(k,q)$, which are examples of **algebraically defined graphs**, that is, graphs whose vertices are coordinate vectors, with two vertices adjacent provided their coordinates satisfy a prescribed set of equations (**adjacency relations**).

More explicitly, each $CD(k,q)$ is a connected component of the graph $D(k,q)$.

Introduction

Graphs $CD(k, q)$ are derived from the graphs $D(k, q)$, which are examples of **algebraically defined graphs**, that is, graphs whose vertices are coordinate vectors, with two vertices adjacent provided their coordinates satisfy a prescribed set of equations (**adjacency relations**).

More explicitly, each $CD(k, q)$ is a connected component of the graph $D(k, q)$.

They exist for every positive integer k and every prime power q.

$$\boldsymbol{— CD(5, q) —}$$

Points: $(p) = (p_1, p_{11}, p_{12}, p_{21}, p_{22}) \subset \mathbb{F}_q^5$
Lines: $[\ell] = [\ell_1, \ell_{11}, \ell_{12}, \ell_{21}, \ell_{22}] \subset \mathbb{F}_q^5$

$$\boldsymbol{— CD(5, q) —}$$

Points: $(p) = (p_1, p_{11}, p_{12}, p_{21}, p_{22}) \subset \mathbb{F}_q^5$

Lines: $[\ell] = [\ell_1, \ell_{11}, \ell_{12}, \ell_{21}, \ell_{22}] \subset \mathbb{F}_q^5$

Adjacency relations:

$$\begin{aligned} p_{11} - \ell_{11} &= p_1 \ell_1 \\ p_{12} - \ell_{12} &= p_1 \ell_{11} \\ p_{21} - \ell_{21} &= p_{11} \ell_1 \\ p_{22} - \ell_{22} &= p_{12} \ell_1 \end{aligned}$$

$$\text{— } \boldsymbol{CD(4,q)} \text{ —}$$

Points: $(p) = (p_1, p_{11}, p_{12}, p_{21}, \cancel{p_{22}}) \subset \mathbb{F}_q^4$

Lines: $[\ell] = [\ell_1, \ell_{11}, \ell_{12}, \ell_{21}, \cancel{\ell_{22}}] \subset \mathbb{F}_q^4$

Adjacency relations:

$$\begin{aligned}
p_{11} - \ell_{11} &= p_1\ell_1 \\
p_{12} - \ell_{12} &= p_1\ell_{11} \\
p_{21} - \ell_{21} &= p_{11}\ell_1 \\
\cancel{p_{22} - \ell_{22}} &= \cancel{p_{12}\ell_1}
\end{aligned}$$

— $\boldsymbol{CD(3,q)}$ —

Points: $(p) = (p_1, p_{11}, p_{12}, \cancel{p_{21}}, \cancel{p_{22}}) \subset \mathbb{F}_q^3$

Lines: $[\ell] = [\ell_1, \ell_{11}, \ell_{12}, \cancel{\ell_{21}}, \cancel{\ell_{22}}] \subset \mathbb{F}_q^3$

Adjacency relations:

$$\begin{aligned} p_{11} - \ell_{11} &= p_1\ell_1 \\ p_{12} - \ell_{12} &= p_1\ell_{11} \\ \cancel{p_{21} - \ell_{21}} &= \cancel{p_{11}\ell_1} \\ \cancel{p_{22} - \ell_{22}} &= \cancel{p_{12}\ell_1} \end{aligned}$$

$$\text{— } \boldsymbol{CD(2,q)} \text{ —}$$

Points: $(p) = (p_1, p_{11}, \cancel{p_{12}}, \cancel{p_{21}}, \cancel{p_{22}}) \subset \mathbb{F}_q^2$

Lines: $[\ell] = [\ell_1, \ell_{11}, \cancel{\ell_{12}}, \cancel{\ell_{21}}, \cancel{\ell_{22}}] \subset \mathbb{F}_q^2$

Adjacency relations:

$$\begin{aligned}
p_{11} - \ell_{11} &= p_1\ell_1 \\
\cancel{p_{12} - \ell_{12}} &= \cancel{p_1\ell_{11}} \\
\cancel{p_{21} - \ell_{21}} &= \cancel{p_{11}\ell_1} \\
\cancel{p_{22} - \ell_{22}} &= \cancel{p_{12}\ell_1}
\end{aligned}$$

Motivation

Even Circuit Theorem (Erdős): Let Γ be a graph with v vertices and e edges, and assume Γ contains no $2k$-cycle. Then

$$\boldsymbol{e \leq O\left(v^{1+\frac{1}{k}}\right)}$$

Motivation

Even Circuit Theorem (Erdős): Let Γ be a graph with v vertices and e edges, and assume Γ contains no $2k$-cycle. Then

$$\boldsymbol{e \leq O\left(v^{1+\frac{1}{k}}\right)}$$

This bound is known to be sharp for only three values of $\boldsymbol{k}$, namely $\boldsymbol{k = 2, 3, 5}$.

Even Circuit Theorem (Erdős): Let Γ be a graph with v vertices and e edges, and assume Γ contains no $2k$-cycle. Then

$$\boldsymbol{e \leq O\left(v^{1+\frac{1}{k}}\right)}$$

This bound is known to be sharp for only three values of $\boldsymbol{k}$, namely $\boldsymbol{k = 2, 3, 5}$.

Problem: How close to this bound can we come for the remaining values of $\boldsymbol{k}$?

Motivation

The GOOD news:

Motivation

The GOOD news:

The cases $\boldsymbol{k = 2, 3, 5}$ are realized by the respective incidence graphs of the generalized triangle of type $\boldsymbol{A_2}$, the generalized quadrangle of type $\boldsymbol{B_2}$, and the generalized hexagon of type $\boldsymbol{G_2}$.

The GOOD news:

The cases $k = 2, 3, 5$ are realized by the respective incidence graphs of the generalized triangle of type A_2, the generalized quadrangle of type B_2, and the generalized hexagon of type G_2.

The BAD news:

The GOOD news:

The cases $k = 2, 3, 5$ are realized by the respective incidence graphs of the generalized triangle of type A_2, the generalized quadrangle of type B_2, and the generalized hexagon of type G_2.

The BAD news:

Theorem (Feit-Higman): A finite thick generalized m-gon exists only for $m \in \{2, 3, 4, 6, 8\}$.

Motivation

The GOOD news:

The cases $k = 2, 3, 5$ are realized by the respective incidence graphs of the generalized triangle of type A_2, the generalized quadrangle of type B_2, and the generalized hexagon of type G_2.

The BAD news:

Theorem (Feit-Higman): A finite thick generalized m-gon exists only for $m \in \{2, 3, 4, 6, 8\}$.

The cases $k = 2, 3, 5$ correspond to $m = 3, 4, 6$ respectively. No other value of m can contribute anything meaningful to our extremal graph theory problem.

Motivation

What was the goal of our research?

Motivation

What was the goal of our research?

To derive families of graphs that simultaneously

Motivation

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,

Motivation

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

Motivation

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

Outline:

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

Outline:

1. Root systems

Motivation

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

Outline:

1. Root systems
2. The Weyl group

Motivation

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

Outline:

1. Root systems
2. The Weyl group
3. Dynkin diagrams and the Cartan matrix

Motivation

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

Outline:

1. Root systems
2. The Weyl group
3. Dynkin diagrams and the Cartan matrix
4. Lie algebras and the upper Borel subalgebra

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

Outline:

1. Root systems
2. The Weyl group
3. Dynkin diagrams and the Cartan matrix
4. Lie algebras and the upper Borel subalgebra
5. Lie groups and finite groups of Lie type

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

Outline:

1. Root systems
2. The Weyl group
3. Dynkin diagrams and the Cartan matrix
4. Lie algebras and the upper Borel subalgebra
5. Lie groups and finite groups of Lie type
6. Rank 2 buildings

Motivation

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

Outline:

1. Root systems
2. The Weyl group
3. Dynkin diagrams and the Cartan matrix
4. Lie algebras and the upper Borel subalgebra
5. Lie groups and finite groups of Lie type
6. Rank 2 buildings
7. Embedding buildings into Lie algebras

Motivation

What was the goal of our research?

To derive families of graphs that simultaneously

(i) mimic the behavior of incidence graphs of generalized m-gons,
(ii) exist for infinitely many values of m.

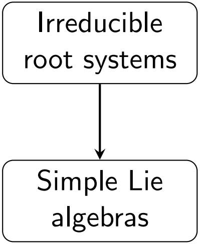
Outline:

1. Root systems
2. The Weyl group
3. Dynkin diagrams and the Cartan matrix
4. Lie algebras and the upper Borel subalgebra
5. Lie groups and finite groups of Lie type
6. Rank 2 buildings
7. Embedding buildings into Lie algebras
8. Objects of type $\widetilde{A}_1$

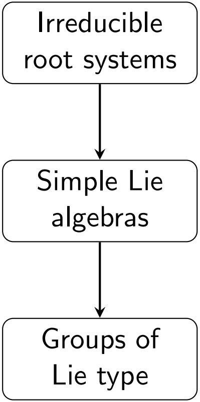
Schematic

Irreducible
root systems

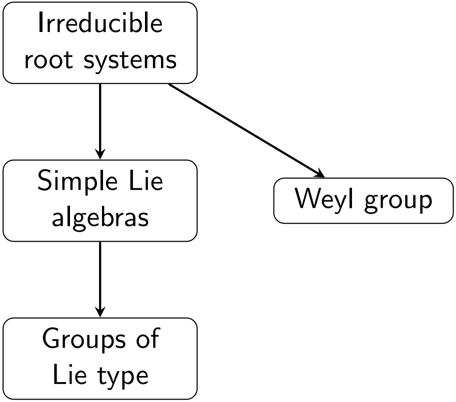
Schematic



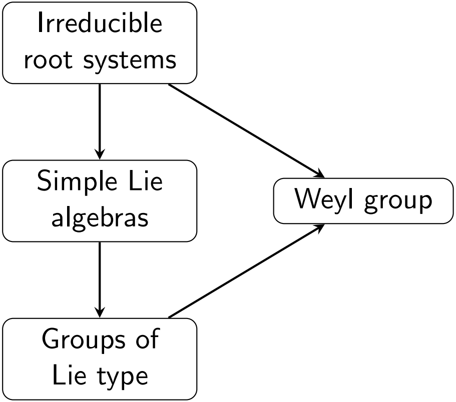
Schematic


Irreducible root systems
Simple Lie algebras
Groups of Lie type

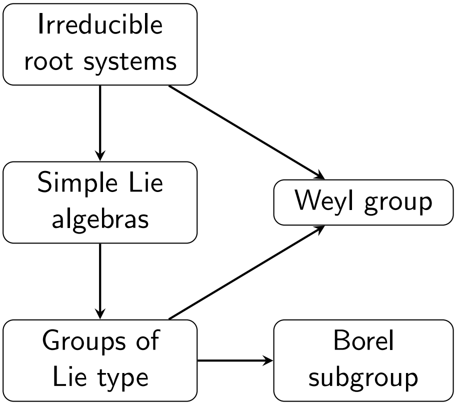
Schematic


Irreducible root systems
Simple Lie algebras
Weyl group
Groups of Lie type

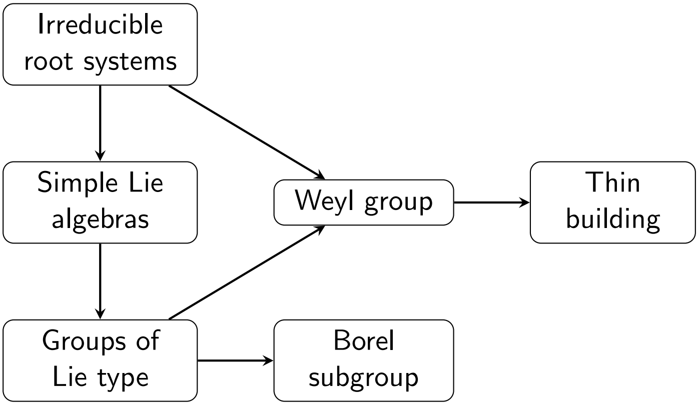
Schematic


Irreducible root systems
Simple Lie algebras
Groups of Lie type
Weyl group

Schematic

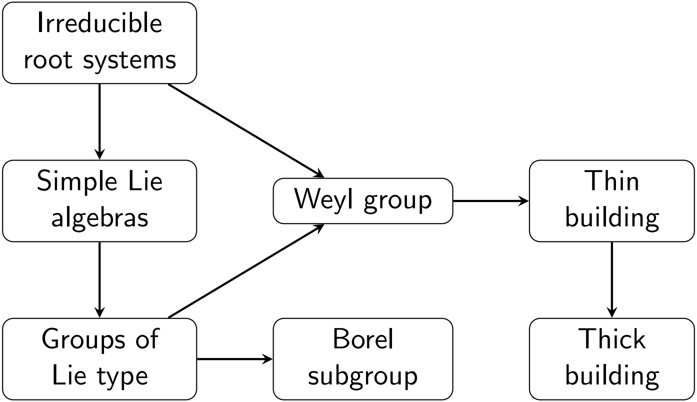

Irreducible root systems
Simple Lie algebras
Weyl group
Groups of Lie type
Borel subgroup

Schematic

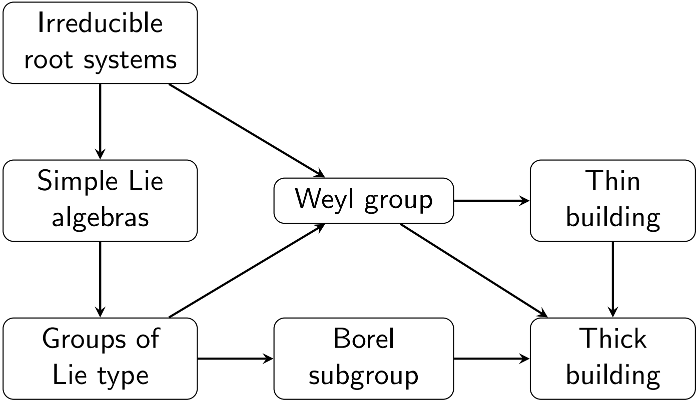


Irreducible root systems
Simple Lie algebras
Groups of Lie type
Weyl group
Thin building
Borel subgroup

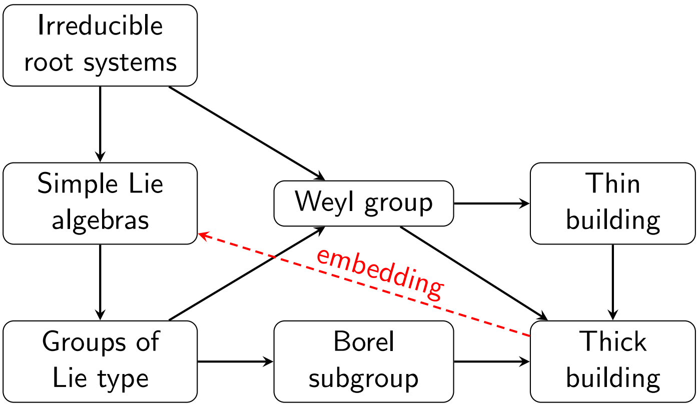
Schematic


Irreducible root systems
Simple Lie algebras
Groups of Lie type
Weyl group
Borel subgroup
Thin building
Thick building

Schematic


Irreducible root systems
Simple Lie algebras
Weyl group
Thin building
Groups of Lie type
Borel subgroup
Thick building

Schematic



Irreducible root systems
Simple Lie algebras
Weyl group
Thin building
embedding
Groups of Lie type
Borel subgroup
Thick building

Let V be a finite-dimensional Euclidean space with standard inner product $(\cdot, \cdot)$.

Let V be a finite-dimensional Euclidean space with standard inner product $(\cdot, \cdot)$.

A **root system** Φ is a set of vectors in V such that

Let V be a finite-dimensional Euclidean space with standard inner product $(\cdot, \cdot)$.

A **root system** Φ is a set of vectors in V such that

1. $|\Phi| < \infty$

Let V be a finite-dimensional Euclidean space with standard inner product $(\cdot, \cdot)$.

A **root system** Φ is a set of vectors in V such that

1. $|\Phi| < \infty$
2. Φ spans V

Let V be a finite-dimensional Euclidean space with standard inner product $(\cdot, \cdot)$.

A **root system** Φ is a set of vectors in V such that

1. $|\Phi| < \infty$
2. Φ spans V
3. For every $r, s \in \Phi$, one has $2\frac{(r,s)}{(r,r)} \in \mathbb{Z}$

Let V be a finite-dimensional Euclidean space with standard inner product $(\cdot,\cdot)$.

A **root system** Φ is a set of vectors in V such that

1. $|\Phi| < \infty$
2. Φ spans V
3. For every $r, s \in \Phi$, one has $2\frac{(r,s)}{(r,r)} \in \mathbb{Z}$
4. For every $r, s \in \Phi$, one has $s - 2\frac{(r,s)}{(r,r)}r \in \Phi$

Let V be a finite-dimensional Euclidean space with standard inner product $(\cdot, \cdot)$.

A **root system** Φ is a set of vectors in V such that

1. $|\Phi| < \infty$
2. Φ spans V
3. For every $r, s \in \Phi$, one has $2\frac{(r,s)}{(r,r)} \in \mathbb{Z}$
4. For every $r, s \in \Phi$, one has $s - 2\frac{(r,s)}{(r,r)}r \in \Phi$
5. For every $r \in \Phi$, one has $\{\alpha r \mid \alpha \in \mathbb{R}\} \cap \Phi = \{r, -r\}$

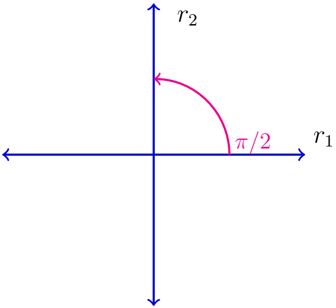
Let V be a finite-dimensional Euclidean space with standard inner product $(\cdot,\cdot)$.

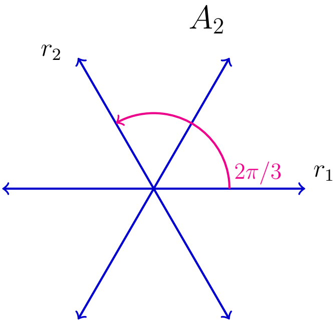
A **root system** Φ is a set of vectors in V such that

1. $|\Phi| < \infty$
2. Φ spans V
3. For every $r, s \in \Phi$, one has $2\frac{(r,s)}{(r,r)} \in \mathbb{Z}$
4. For every $r, s \in \Phi$, one has $s - 2\frac{(r,s)}{(r,r)}r \in \Phi$
5. For every $r \in \Phi$, one has $\{\alpha r \mid \alpha \in \mathbb{R}\} \cap \Phi = \{r, -r\}$

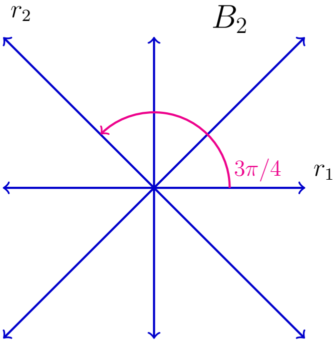
(Hence $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^- = -\Phi^+$.)

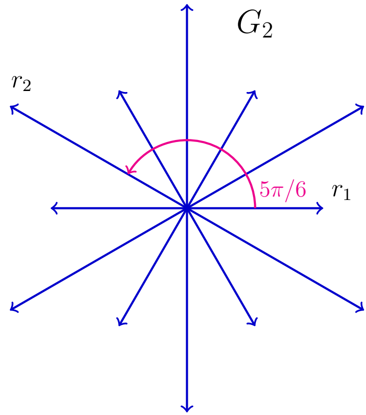
$A_1 \times A_1$





A_2
r_2
$2\pi/3$
r_1


r_2
B_2
$3\pi/4$
r_1


G_2
r_2
$5\pi/6$
r_1

Let Φ be a root system with **fundamental roots** $r_1, r_2, \ldots, r_n$.

Let Φ be a root system with **fundamental roots** $r_1, r_2, \ldots, r_n$.

Set $\Pi = \{r_1, r_2, \ldots, r_n\}$ (**fundamental basis**).

Let Φ be a root system with **fundamental roots** $r_1, r_2, \ldots, r_n$.

Set $\Pi = \{r_1, r_2, \ldots, r_n\}$ (**fundamental basis**).

For each $r_i \in \Pi$, denote by w_i the reflection in the hyperplane in $V = \mathbb{R}^n$ orthogonal to r_i, that is,

$$\boldsymbol{w_i(s) = s - 2\frac{(r_i,s)}{(r_i,r_i)}r_i}$$

Let Φ be a root system with **fundamental roots** $r_1, r_2, \ldots, r_n$.

Set $\Pi = \{r_1, r_2, \ldots, r_n\}$ (**fundamental basis**).

For each $r_i \in \Pi$, denote by w_i the reflection in the hyperplane in $V = \mathbb{R}^n$ orthogonal to r_i, that is,

$$\boldsymbol{w_i(s) = s - 2\frac{(r_i,s)}{(r_i,r_i)}r_i} \in \boldsymbol{\Phi} \text{ (Axiom 4 of a root system)}$$

Let Φ be a root system with **fundamental roots** $r_1, r_2, \ldots, r_n$.

Set $\Pi = \{r_1, r_2, \ldots, r_n\}$ (**fundamental basis**).

For each $r_i \in \Pi$, denote by w_i the reflection in the hyperplane in $V = \mathbb{R}^n$ orthogonal to r_i, that is,

$$\boldsymbol{w_i(s) = s - 2\frac{(r_i,s)}{(r_i,r_i)} r_i} \in \boldsymbol{\Phi} \text{ (Axiom 4 of a root system)}$$

Thus each w_i permutes the roots of Φ.

Let Φ be a root system with **fundamental roots** $r_1, r_2, \ldots, r_n$.

Set $\Pi = \{r_1, r_2, \ldots, r_n\}$ (**fundamental basis**).

For each $r_i \in \Pi$, denote by w_i the reflection in the hyperplane in $V = \mathbb{R}^n$ orthogonal to r_i, that is,

$$\boldsymbol{w_i(s) = s - 2\frac{(r_i,s)}{(r_i,r_i)}r_i} \in \boldsymbol{\Phi} \text{ (Axiom 4 of a root system)}$$

Thus each w_i permutes the roots of Φ.

We call w_i a **fundamental reflection**.

Let Φ be a root system with **fundamental roots** $r_1, r_2, \ldots, r_n$.

Set $\Pi = \{r_1, r_2, \ldots, r_n\}$ (**fundamental basis**).

For each $r_i \in \Pi$, denote by w_i the reflection in the hyperplane in $V = \mathbb{R}^n$ orthogonal to r_i, that is,

$$\boldsymbol{w_i(s) = s - 2\frac{(r_i, s)}{(r_i, r_i)} r_i} \in \boldsymbol{\Phi} \text{ (Axiom 4 of a root system)}$$

Thus each w_i permutes the roots of Φ.

We call w_i a **fundamental reflection**.

The **Weyl group** is generated by all fundamental reflections, i.e.,

$$W = \langle w_1, w_2, \ldots, w_n \rangle$$

Let $\measuredangle r_i r_j$ denote the angle between $r_i, r_j \in \Pi$.

Let $\measuredangle r_i r_j$ denote the angle between $r_i, r_j \in \Pi$.

Then $\measuredangle r_i r_j = \left(\frac{m-1}{m}\right)\pi$ where $m \in \{2, 3, 4, 6\}$.

Let $\measuredangle r_i r_j$ denote the angle between $r_i, r_j \in \Pi$.

Then $\measuredangle r_i r_j = \left(\frac{m-1}{m}\right)\pi$ where $m \in \{2,3,4,6\}$.

In fact, m is the order $o(w_i w_j)$ of the rotation $w_i w_j$ in the plane determined by r_i, r_j.

Let $\measuredangle r_i r_j$ denote the angle between $r_i, r_j \in \Pi$.

Then $\measuredangle r_i r_j = \left(\frac{m-1}{m}\right)\pi$ where $m \in \{2, 3, 4, 6\}$.

In fact, m is the order $o(w_i w_j)$ of the rotation $w_i w_j$ in the plane determined by r_i, r_j.

This implies that the Weyl group of every rank 2 root system is a dihedral group of order $2m$:

Let $\measuredangle r_i r_j$ denote the angle between $r_i, r_j \in \Pi$.

Then $\measuredangle r_i r_j = \left(\frac{m-1}{m}\right)\pi$ where $m \in \{2, 3, 4, 6\}$.

In fact, m is the order $o(w_i w_j)$ of the rotation $w_i w_j$ in the plane determined by r_i, r_j.

This implies that the Weyl group of every rank 2 root system is a dihedral group of order $2m$:

- $W(A_1 \times A_1) \cong D_4$ $(m = 2)$

Let $\measuredangle r_i r_j$ denote the angle between $r_i, r_j \in \Pi$.

Then $\measuredangle r_i r_j = \left(\frac{m-1}{m}\right)\pi$ where $m \in \{2, 3, 4, 6\}$.

In fact, m is the order $o(w_i w_j)$ of the rotation $w_i w_j$ in the plane determined by r_i, r_j.

This implies that the Weyl group of every rank 2 root system is a dihedral group of order $2m$:

- $W(A_1 \times A_1) \cong D_4 \ (m = 2)$
- $W(A_2) \cong D_6 \ (m = 3)$

Let $\measuredangle r_i r_j$ denote the angle between $r_i, r_j \in \Pi$.

Then $\measuredangle r_i r_j = \left(\frac{m-1}{m}\right)\pi$ where $m \in \{2, 3, 4, 6\}$.

In fact, m is the order $o(w_i w_j)$ of the rotation $w_i w_j$ in the plane determined by r_i, r_j.

This implies that the Weyl group of every rank 2 root system is a dihedral group of order $2m$:

- $W(A_1 \times A_1) \cong D_4$ $(m = 2)$
- $W(A_2) \cong D_6$ $(m = 3)$
- $W(B_2) \cong D_8$ $(m = 4)$

Let $\measuredangle r_i r_j$ denote the angle between $r_i, r_j \in \Pi$.

Then $\measuredangle r_i r_j = \left(\frac{m-1}{m}\right)\pi$ where $m \in \{2,3,4,6\}$.

In fact, m is the order $o(w_i w_j)$ of the rotation $w_i w_j$ in the plane determined by r_i, r_j.

This implies that the Weyl group of every rank 2 root system is a dihedral group of order $2m$:

- $W(A_1 \times A_1) \cong D_4$ $(m = 2)$
- $W(A_2) \cong D_6$ $(m = 3)$
- $W(B_2) \cong D_8$ $(m = 4)$
- $W(G_2) \cong D_{12}$ $(m = 6)$

A **Dynkin diagram** consists of nodes and weighted (directed) edges between pairs of nodes.

A **Dynkin diagram** consists of nodes and weighted (directed) edges between pairs of nodes.

It is a useful device for codifying a root system:

A **Dynkin diagram** consists of nodes and weighted (directed) edges between pairs of nodes.

It is a useful device for codifying a root system:

$$\text{Nodes of diagram} \longleftrightarrow \text{fundamental roots}$$

edge weights determine angles between pairs of roots

A **Dynkin diagram** consists of nodes and weighted (directed) edges between pairs of nodes.

It is a useful device for codifying a root system:

$$\text{Nodes of diagram} \longleftrightarrow \text{fundamental roots}$$

edge weights determine angles between pairs of roots

It also codifies the action of the Weyl group on a root system:

A **Dynkin diagram** consists of nodes and weighted (directed) edges between pairs of nodes.

It is a useful device for codifying a root system:

Nodes of diagram $\longleftrightarrow$ fundamental roots

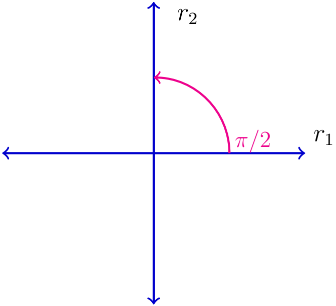
edge weights determine angles between pairs of roots

It also codifies the action of the Weyl group on a root system:

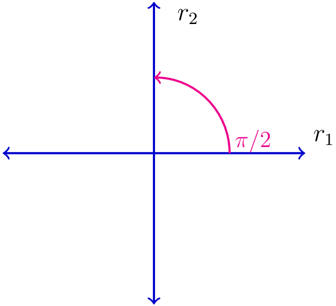
Nodes of diagram $\longleftrightarrow$ fundamental reflections

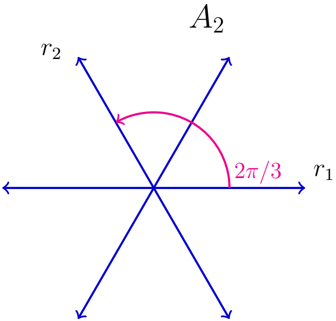
edge weights determine orders of products of pairs of reflections

$A_1 \times A_1$



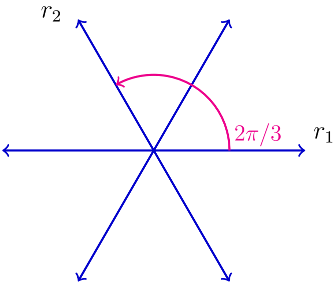
$$A_1 \times A_1$$

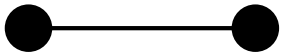



A_2
r_2
$2\pi/3$
r_1

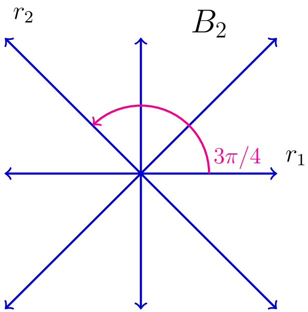
Dynkin diagrams
rank 2 case

A_2

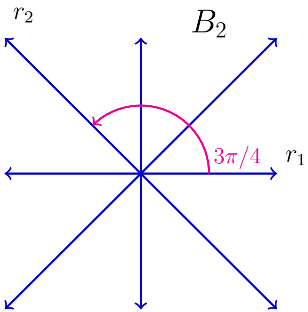


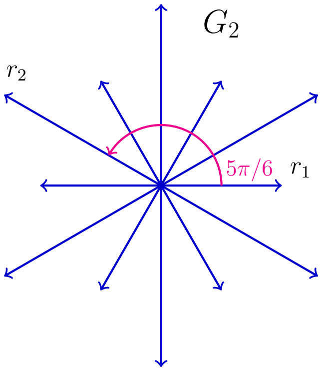


Dynkin diagrams
rank 2 case



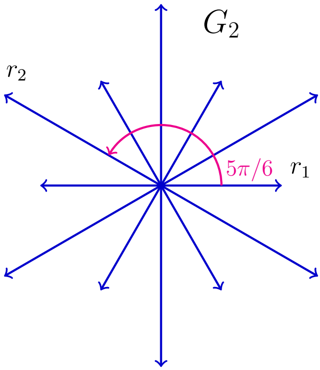
Dynkin diagrams
rank 2 case




G_2
r_2
$5\pi/6$
r_1

Dynkin diagrams

rank 2 case



The Cartan matrix
another codifying device

Recall that for any pair of roots $r, s \in \Phi$, one has

$$\mathbf{2\frac{(r,s)}{(r,r)}} \in \mathbb{Z} \quad \text{(Axiom 3 of a root system)}$$

Recall that for any pair of roots $r, s \in \Phi$, one has

$$\mathbf{2}\frac{(r,s)}{(r,r)} \in \mathbb{Z} \quad \text{(Axiom 3 of a root system)}$$

For fundamental roots $r_i, r_j \in \Pi$, we may express this as

$$A_{ij} = 2\frac{(r_i,r_j)}{(r_i,r_i)} \in \mathbb{Z}.$$

Recall that for any pair of roots $r, s \in \Phi$, one has

$$\mathbf{2}\frac{(\boldsymbol{r},\boldsymbol{s})}{(\boldsymbol{r},\boldsymbol{r})} \in \mathbb{Z} \quad \text{(Axiom 3 of a root system)}$$

For fundamental roots $r_i, r_j \in \Pi$, we may express this as

$$\boldsymbol{A_{ij}} = \mathbf{2}\frac{(\boldsymbol{r_i},\boldsymbol{r_j})}{(\boldsymbol{r_i},\boldsymbol{r_i})} \in \mathbb{Z}.$$

We now define the **Cartan matrix** for Φ by $\boldsymbol{A}(\boldsymbol{\Phi}) = (\boldsymbol{A_{ij}})$.

Recall that for any pair of roots $r, s \in \Phi$, one has

$$\mathbf{2}\frac{(\boldsymbol{r},\boldsymbol{s})}{(\boldsymbol{r},\boldsymbol{r})} \in \mathbb{Z}$$ (Axiom 3 of a root system)

For fundamental roots $r_i, r_j \in \Pi$, we may express this as

$$\boldsymbol{A_{ij}} = \mathbf{2}\frac{(\boldsymbol{r_i},\boldsymbol{r_j})}{(\boldsymbol{r_i},\boldsymbol{r_i})} \in \mathbb{Z}.$$

We now define the **Cartan matrix** for Φ by $\boldsymbol{A(\Phi) = (A_{ij})}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$$A_1 \times A_1 \qquad A_2 \qquad B_2 \qquad G_2$$

The Cartan matrix
another codifying device

Recall that for any pair of roots $r, s \in \Phi$, one has

$$\boldsymbol{2\frac{(r,s)}{(r,r)}} \in \mathbb{Z} \quad \text{(Axiom 3 of a root system)}$$

For fundamental roots $r_i, r_j \in \Pi$, we may express this as

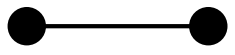
$$\boldsymbol{A_{ij} = 2\frac{(r_i,r_j)}{(r_i,r_i)}} \in \mathbb{Z}.$$

We now define the **Cartan matrix** for Φ by $\boldsymbol{A(\Phi) = (A_{ij})}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

$A_1 \times A_1$

A_2



B_2

G_2

A **Lie algebra** is a vector space $\mathfrak{L}$ over some field $\mathbb{F}$, endowed with a binary operation $[\,\cdot\,,\cdot\,] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ (**Lie product**) which is bilinear, anticommutative, and satisfies the Jacobi identity:

$$[\alpha,[\beta,\gamma]] + [\beta,[\gamma,\alpha]] + [\gamma,[\alpha,\beta]] = 0, \;\; \forall \alpha,\beta,\gamma \in \mathfrak{L}$$

A **Lie algebra** is a vector space $\mathfrak{L}$ over some field $\mathbb{F}$, endowed with a binary operation $[\,\cdot\,,\cdot\,] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ (**Lie product**) which is bilinear, anticommutative, and satisfies the Jacobi identity:

$$[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] = 0, \;\; \forall \alpha, \beta, \gamma \in \mathfrak{L}$$

Lie algebras are examples of non-associative graded algebras.

A **Lie algebra** is a vector space $\mathfrak{L}$ over some field $\mathbb{F}$, endowed with a binary operation $[\,\cdot\,,\cdot\,] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ (**Lie product**) which is bilinear, anticommutative, and satisfies the Jacobi identity:

$$[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] = 0, \ \ \forall \alpha, \beta, \gamma \in \mathfrak{L}$$

Lie algebras are examples of non-associative graded algebras.

Subject to a fixed choice of root system and field, one obtains a unique semisimple Lie algebra.

Let $\mathfrak{H}$ be a self-normalizing nilpotent subalgebra of $\mathfrak{L}$. We call $\mathfrak{H}$ a **Cartan subalgebra** of $\mathfrak{L}$.

Let $\mathfrak{H}$ be a self-normalizing nilpotent subalgebra of $\mathfrak{L}$. We call $\mathfrak{H}$ a **Cartan subalgebra** of $\mathfrak{L}$.

$$\mathfrak{L} = \mathfrak{H} \bigoplus_{r \in \Phi} \mathfrak{L}_r \quad (\textbf{Cartan decomposition})$$

where each root space $\mathfrak{L}_r$ is an $\mathfrak{H}$-invariant subspace of $\mathfrak{L}$, i.e., $[\mathfrak{H}, \mathfrak{L}_r] \subseteq \mathfrak{L}_r$.

Let $\mathfrak{H}$ be a self-normalizing nilpotent subalgebra of $\mathfrak{L}$. We call $\mathfrak{H}$ a **Cartan subalgebra** of $\mathfrak{L}$.

$$\mathfrak{L} = \mathfrak{H} \bigoplus_{r\in\Phi} \mathfrak{L}_r \quad (\textbf{Cartan decomposition})$$

where each root space $\mathfrak{L}_r$ is an $\mathfrak{H}$-invariant subspace of $\mathfrak{L}$, i.e., $[\mathfrak{H}, \mathfrak{L}_r] \subseteq \mathfrak{L}_r$.

If $\mathfrak{L}$ arises from a root system, then each $\mathfrak{L}_r$ is one-dimensional. We write $\mathfrak{L}_r = \langle e_r \rangle$ and refer to e_r as a **root vector**.

Let $\mathfrak{H}$ be a self-normalizing nilpotent subalgebra of $\mathfrak{L}$. We call $\mathfrak{H}$ a **Cartan subalgebra** of $\mathfrak{L}$.

$$\mathfrak{L} = \mathfrak{H} \bigoplus_{r \in \Phi} \mathfrak{L}_r \quad (\textbf{Cartan decomposition})$$

where each root space $\mathfrak{L}_r$ is an $\mathfrak{H}$-invariant subspace of $\mathfrak{L}$, i.e., $[\mathfrak{H}, \mathfrak{L}_r] \subseteq \mathfrak{L}_r$.

If $\mathfrak{L}$ arises from a root system, then each $\mathfrak{L}_r$ is one-dimensional. We write $\mathfrak{L}_r = \langle e_r \rangle$ and refer to e_r as a **root vector**.

For each $r \in \Phi$, one has $[h, e_r] = r(h)e_r$, $h \in \mathfrak{H}$,
i.e., each root $r \in \Phi$ is a linear functional $r : \mathfrak{H} \to \mathbb{R}$.

Let $\mathfrak{H}$ be a self-normalizing nilpotent subalgebra of $\mathfrak{L}$. We call $\mathfrak{H}$ a **Cartan subalgebra** of $\mathfrak{L}$.

$$\mathfrak{L} = \mathfrak{H} \bigoplus_{r\in\Phi} \mathfrak{L}_r \quad (\textbf{Cartan decomposition})$$

where each root space $\mathfrak{L}_r$ is an $\mathfrak{H}$-invariant subspace of $\mathfrak{L}$, i.e., $[\mathfrak{H}, \mathfrak{L}_r] \subseteq \mathfrak{L}_r$.

If $\mathfrak{L}$ arises from a root system, then each $\mathfrak{L}_r$ is one-dimensional. We write $\mathfrak{L}_r = \langle e_r \rangle$ and refer to e_r as a **root vector**.

For each $r \in \Phi$, one has $[h, e_r] = r(h)e_r$, $h \in \mathfrak{H}$,
i.e., each root $r \in \Phi$ is a linear functional $r : \mathfrak{H} \to \mathbb{R}$.

This gives $\Phi \hookrightarrow \mathfrak{H}^*$, therefore $\boldsymbol{\Phi^* \hookrightarrow \mathfrak{H}}$.

Recall that $\Phi = \Phi^+ \cup \Phi^-$.

Recall that $\Phi = \Phi^+ \cup \Phi^-$.

It follows that

$$\bigoplus_{r\in\Phi} \mathfrak{L}_r = \mathfrak{L_r}^+ \bigoplus \mathfrak{L_r}^-$$

where

$$\mathfrak{L_r}^+ = \bigoplus_{r\in\Phi^+} \mathfrak{L}_r \quad (\textbf{positive root space})$$

$$\mathfrak{L_r}^- = \bigoplus_{r\in\Phi^-} \mathfrak{L}_r \quad (\textbf{negative root space})$$

Recall that $\Phi = \Phi^+ \cup \Phi^-$.

It follows that

$$\bigoplus_{r \in \Phi} \mathfrak{L}_r = \mathfrak{L_r}^+ \bigoplus \mathfrak{L_r}^-$$

where

$$\mathfrak{L_r}^+ = \bigoplus_{r \in \Phi^+} \mathfrak{L}_r \quad (\textbf{positive root space})$$

$$\mathfrak{L_r}^- = \bigoplus_{r \in \Phi^-} \mathfrak{L}_r \quad (\textbf{negative root space})$$

If we now append $\mathfrak{H}$ to each of these subspaces, we obtain

$$\mathfrak{L}^U = \mathfrak{H} \bigoplus \mathfrak{L}_r^+ \quad (\textbf{upper Borel subalgebra})$$

$$\mathfrak{L}^L = \mathfrak{H} \bigoplus \mathfrak{L}_r^- \quad (\textbf{lower Borel subalgebra})$$

Recall that $\Phi = \Phi^+ \cup \Phi^-$.

It follows that

$$\bigoplus_{r\in\Phi} \mathfrak{L}_r = \mathfrak{L}_{\mathfrak{r}}{}^+ \bigoplus \mathfrak{L}_{\mathfrak{r}}{}^-$$

where

$$\mathfrak{L}_{\mathfrak{r}}{}^+ = \bigoplus_{r\in\Phi^+} \mathfrak{L}_r \quad (\textbf{positive root space})$$

$$\mathfrak{L}_{\mathfrak{r}}{}^- = \bigoplus_{r\in\Phi^-} \mathfrak{L}_r \quad (\textbf{negative root space})$$

If we now append $\mathfrak{H}$ to each of these subspaces, we obtain

$$\mathfrak{L}^U = \mathfrak{H} \bigoplus \mathfrak{L}_r^+ \quad (\textbf{upper Borel subalgebra})$$

$$\mathfrak{L}^L = \mathfrak{H} \bigoplus \mathfrak{L}_r^- \quad (\textbf{lower Borel subalgebra})$$

We are interested in the upper Borel subalgebra $\boldsymbol{\mathfrak{L}^U = \mathfrak{H} \bigoplus \mathfrak{L}_r^+}$.

EXAMPLE. We construct the complex Lie algebra $\mathfrak{L}$ of type A_2 and identify its upper Borel subalgebra $\mathfrak{L}^U$.

EXAMPLE. We construct the complex Lie algebra $\mathfrak{L}$ of type A_2 and identify its upper Borel subalgebra $\mathfrak{L}^U$.

First we determine the root system of type A_2:

$$\Phi = \{r_1, r_2, r_1 + r_2, -r_1, -r_2, -r_1 - r_2\}$$

EXAMPLE. We construct the complex Lie algebra $\mathfrak{L}$ of type A_2 and identify its upper Borel subalgebra $\mathfrak{L}^U$.

First we determine the root system of type A_2:

$$\Phi = \{r_1, r_2, r_1 + r_2, -r_1, -r_2, -r_1 - r_2\}$$

Let $\Pi^* = \{r_1^*, r_2^*\}$ be the dual basis of the fundamental basis $\Pi = \{r_1, r_2\}$.

EXAMPLE. We construct the complex Lie algebra $\mathfrak{L}$ of type A_2 and identify its upper Borel subalgebra $\mathfrak{L}^U$.

First we determine the root system of type A_2:

$$\Phi = \{r_1, r_2, r_1 + r_2, -r_1, -r_2, -r_1 - r_2\}$$

Let $\Pi^* = \{r_1^*, r_2^*\}$ be the dual basis of the fundamental basis $\Pi = \{r_1, r_2\}$.

Then Π^* is a basis for $\langle \Phi^* \rangle = \mathfrak{H}$, and we accordingly obtain the following canonical basis for $\mathfrak{L}$:

EXAMPLE. We construct the complex Lie algebra $\mathfrak{L}$ of type A_2 and identify its upper Borel subalgebra $\mathfrak{L}^U$.

First we determine the root system of type A_2:

$$\Phi = \{r_1, r_2, r_1 + r_2, -r_1, -r_2, -r_1 - r_2\}$$

Let $\Pi^* = \{r_1^*, r_2^*\}$ be the dual basis of the fundamental basis $\Pi = \{r_1, r_2\}$.

Then Π^* is a basis for $\langle \Phi^* \rangle = \mathfrak{H}$, and we accordingly obtain the following canonical basis for $\mathfrak{L}$:

$$\{\underbrace{r_1^*, r_2^*}_{\mathfrak{H}}, \underbrace{e_{r_1}, e_{r_2}, e_{r_1+r_2}}_{\mathfrak{L}^+}, \underbrace{e_{-r_1}, e_{-r_2}, e_{-r_1-r_2}}_{\mathfrak{L}^-}\}$$

We may choose our embedding $\Pi^* \hookrightarrow \mathfrak{H}$ as follows:

$$r_1^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad r_2^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We may choose our embedding $\Pi^* \hookrightarrow \mathfrak{H}$ as follows:

$$r_1^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad r_2^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Similarly, we obtain matrix representations of the six root vectors:

We may choose our embedding $\Pi^* \hookrightarrow \mathfrak{H}$ as follows:

$$r_1^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad r_2^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Similarly, we obtain matrix representations of the six root vectors:

$$e_{r_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_{r_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad e_{r_1+r_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We may choose our embedding $\Pi^* \hookrightarrow \mathfrak{H}$ as follows:

$$r_1^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad r_2^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Similarly, we obtain matrix representations of the six root vectors:

$$e_{r_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_{r_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad e_{r_1+r_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_{-r_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_{-r_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad e_{-r_1-r_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

A basis for $\mathfrak{L}^U = \mathfrak{H} \oplus \mathfrak{L}^+$ is

$$\{\underbrace{r_1^*, r_2^*}_{\mathfrak{H}}, \underbrace{e_{r_1}, e_{r_2}, e_{r_1+r_2}}_{\mathfrak{L}^+}\}$$

A basis for $\mathfrak{L}^U = \mathfrak{H} \oplus \mathfrak{L}^+$ is

$$\{\underbrace{r_1^*, r_2^*}_{\mathfrak{H}}, \underbrace{e_{r_1}, e_{r_2}, e_{r_1+r_2}}_{\mathfrak{L}^+}\}$$

Subject to our matrix representation, this becomes

$$\mathfrak{L}^U = \left\{ \begin{pmatrix} a & c & d \\ 0 & b-a & e \\ 0 & 0 & -b \end{pmatrix} \middle| \, a, b, c, d, e \in \mathbb{C} \right\}$$

□

Given a complex simple Lie algebra $\mathfrak{L}$, for each $x \in \mathfrak{L}$ we define the exponentiation map

$$\exp(\operatorname{ad} x) = \sum_{k=0}^{\infty} \frac{(\operatorname{ad}\, x)^k}{k!}$$

where

$$\operatorname{ad} x : \mathfrak{L} \to \mathfrak{L} \quad (\operatorname{ad} x : y \mapsto [x, y])$$

Given a complex simple Lie algebra $\mathfrak{L}$, for each $x \in \mathfrak{L}$ we define the exponentiation map

$$\exp{(\mathrm{ad}\, x)} = \sum_{k=0}^{\infty} \frac{(\mathrm{ad}\ x)^k}{k!}$$

where

$$\mathrm{ad}\, x : \mathfrak{L} \to \mathfrak{L} \quad (\mathrm{ad}\, x : y \mapsto [x, y])$$

Note: $\exp{(\mathrm{ad}\, x)}\ \exp{(\mathrm{ad}\, y)} = \exp{(\mathrm{ad}\, [x, y])}$.

Given a complex simple Lie algebra $\mathfrak{L}$, for each $x \in \mathfrak{L}$ we define the exponentiation map

$$\exp(\operatorname{ad} x) = \sum_{k=0}^{\infty} \frac{(\operatorname{ad} x)^k}{k!}$$

where

$$\operatorname{ad} x : \mathfrak{L} \to \mathfrak{L} \quad (\operatorname{ad} x : y \mapsto [x, y])$$

Note: $\exp(\operatorname{ad} x)\,\exp(\operatorname{ad} y) = \exp(\operatorname{ad}[x, y])$.

As $\operatorname{ad} x$ is a nilpotent derivation, we get that each $\exp(\operatorname{ad} x)$ is an inner automorphism of $\mathfrak{L}$.

Given a complex simple Lie algebra $\mathfrak{L}$, for each $x \in \mathfrak{L}$ we define the exponentiation map

$$\exp(\operatorname{ad} x) = \sum_{k=0}^{\infty} \frac{(\operatorname{ad} x)^k}{k!}$$

where

$$\operatorname{ad} x : \mathfrak{L} \to \mathfrak{L} \quad (\operatorname{ad} x : y \mapsto [x, y])$$

Note: $\exp(\operatorname{ad} x)\,\exp(\operatorname{ad} y) = \exp(\operatorname{ad}[x, y])$.

As $\operatorname{ad} x$ is a nilpotent derivation, we get that each $\exp(\operatorname{ad} x)$ is an inner automorphism of $\mathfrak{L}$.

We now define the complex Lie group:

$$\boldsymbol{G = \{\exp(\operatorname{ad} x) \mid x \in \mathfrak{L}\}}$$

Chevalley constructed a basis (**Chevalley basis**) for the universal enveloping algebra of every complex simple Lie algebra with the property that all structure constants of the enveloping algebra are integral with respect to the basis.

Chevalley constructed a basis (**Chevalley basis**) for the universal enveloping algebra of every complex simple Lie algebra with the property that all structure constants of the enveloping algebra are integral with respect to the basis.

This allows the corresponding algebraic groups to be defined over $\mathbb{Z}$, which enables their range of definition to be extended to finite fields.

Chevalley constructed a basis (**Chevalley basis**) for the universal enveloping algebra of every complex simple Lie algebra with the property that all structure constants of the enveloping algebra are integral with respect to the basis.

This allows the corresponding algebraic groups to be defined over $\mathbb{Z}$, which enables their range of definition to be extended to finite fields.

The resulting finite simple groups are termed **Chevalley groups** in his honor.

Lie type	group	discoverer

Lie type	group	discoverer
$A_n(q)$	$L_{n+1}(q)$	Dickson
$B_n(q)$	$O_{2n+1}(q)$	Dickson
$C_n(q)$	$PSp_{2n}(q)$	Dickson
$D_n(q)$	$P\Omega^+_{2n}(q)$	Dickson

$\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\}$ classical*

Lie type	group	discoverer	
$A_n(q)$	$L_{n+1}(q)$	Dickson	classical*
$B_n(q)$	$O_{2n+1}(q)$	Dickson	classical*
$C_n(q)$	$PSp_{2n}(q)$	Dickson	classical*
$D_n(q)$	$P\Omega^+_{2n}(q)$	Dickson	classical*
$G_2(q)$		Dickson	exceptional
$F_4(q)$		Chevalley	exceptional
$E_6(q)$		Dickson	exceptional
$E_7(q)$		Chevalley	exceptional
$E_8(q)$		Chevalley	exceptional

* The case "q prime" was treated by C. Jordan.

Lie type	group	discoverer

Lie type	group	discoverer	
${}^2A_n(q)$	$U_{n+1}(q)$	Steinberg	classical
${}^2D_n(q)$	$P\Omega^-_{2n}(q)$	Steinberg	

Lie type	group	discoverer	
$^2A_n(q)$	$U_{n+1}(q)$	Steinberg	classical
$^2D_n(q)$	$P\Omega^-_{2n}(q)$	Steinberg	
$^2E_6(q)$		Steinberg*	exceptional
$^3D_4(q)$		Steinberg	
$^2B_2(2^{2m+1})$		Suzuki	
$^2G_2(3^{2m+1})$		Ree	
$^2F_4(2^{2m+1})$		Ree	

* The family $^2E_6(q)$ was discovered independently by J. Tits.

Among these 16 infinite families of groups of Lie type, four such families have no complex analogues:

Among these 16 infinite families of groups of Lie type, four such families have no complex analogues:

$$^3D_4(q),\ ^2B_2(2^{2m+1}),\ ^2G_2(3^{2m+1}),\ ^2F_4(2^{2m+1})$$

Among these 16 infinite families of groups of Lie type, four such families have no complex analogues:

$$^3D_4(q),\ ^2B_2(2^{2m+1}),\ ^2G_2(3^{2m+1}),\ ^2F_4(2^{2m+1})$$

(1) The diagram D_4 admits an order 3 automorphism, however existence of $^3D_4(q)$ requires that the field $\mathbb{F}_q$ be a cubic extension of a smaller field. This precludes the existence of a complex Lie group of type $^3D_4$ over $\mathbb{C}$.

Among these 16 infinite families of groups of Lie type, four such families have no complex analogues:

$$^3D_4(q),\ ^2B_2(2^{2m+1}),\ ^2G_2(3^{2m+1}),\ ^2F_4(2^{2m+1})$$

(1) The diagram D_4 admits an order 3 automorphism, however existence of $^3D_4(q)$ requires that the field $\mathbb{F}_q$ be a cubic extension of a smaller field. This precludes the existence of a complex Lie group of type $^3D_4$ over $\mathbb{C}$.

(2) The diagrams B_2, G_2, F_4 each admit an order 2 automorphism that interchanges long and short roots. The existence of $^2B_2(q)$, $^2G_2(q)$, $^2F_4(q)$ therefore requires that $B_2(q)$, $G_2(q)$, $F_4(q)$ admit graph-field automorphisms that preserve root length. This occurs only for the fields specifed above, and certainly not for $\mathbb{C}$.

Let G be a finite group of Lie type of rank n over $\mathbb{F}_q$, $q = p^a$.

Let G be a finite group of Lie type of rank n over $\mathbb{F}_q$, $q = p^a$.

Fix a Sylow p-subgroup U of G (**unipotent subgroup**).

Let G be a finite group of Lie type of rank n over $\mathbb{F}_q$, $q = p^a$.

Fix a Sylow p-subgroup U of G (**unipotent subgroup**).

Let $B = N_G(U)$ (**Borel subgroup**).

Let G be a finite group of Lie type of rank n over $\mathbb{F}_q$, $q = p^a$.

Fix a Sylow p-subgroup U of G (**unipotent subgroup**).

Let $B = N_G(U)$ (**Borel subgroup**).

Then the full lattice of subgroups $\{P \mid B \leq P \leq G\}$ is isomorphic to the lattice of all subsets of an n-element set.

Let G be a finite group of Lie type of rank n over $\mathbb{F}_q$, $q = p^a$.

Fix a Sylow p-subgroup U of G (**unipotent subgroup**).

Let $B = N_G(U)$ (**Borel subgroup**).

Then the full lattice of subgroups $\{P \mid B \leq P \leq G\}$ is isomorphic to the lattice of all subsets of an n-element set.

The $2^n - 1$ proper subgroups in this lattice are called **parabolic subgroups** of G. Of these, n are maximal subgroups of G. We denote these as $P_1, P_2, \ldots, P_n$ (**maximal parabolics**).

Let G be a finite group of Lie type, and let $B = N_G(U)$ be a fixed Borel subgroup of G.

Let G be a finite group of Lie type, and let $B = N_G(U)$ be a fixed Borel subgroup of G.

As B splits over U, there exists a subgroup $T \leq B$ with $T \cong B/U$. We call T a **maximal torus**.

Let G be a finite group of Lie type, and let $B = N_G(U)$ be a fixed Borel subgroup of G.

As B splits over U, there exists a subgroup $T \leq B$ with $T \cong B/U$. We call T a **maximal torus**.

The Weyl group $W = W(G)$ now appears as the quotient

$$\boldsymbol{W \cong N_G(T)/T}$$

Let G be a finite group of Lie type, and let $B = N_G(U)$ be a fixed Borel subgroup of G.

As B splits over U, there exists a subgroup $T \leq B$ with $T \cong B/U$. We call T a **maximal torus**.

The Weyl group $W = W(G)$ now appears as the quotient

$$\boldsymbol{W \cong N_G(T)/T}$$

As $N_G(T)$ need not split over T, W need not be a subgroup of G.

EXAMPLE. We illustrate the case $A_2(q) = L_3(q)$ in detail.

EXAMPLE. We illustrate the case $A_2(q) = L_3(q)$ in detail.

$$
\begin{tikzcd}
 & G \arrow[dl, no head] \arrow[dr, no head] & & & & \emptyset \arrow[dl, no head] \arrow[dr, no head] & \\
P_1 \arrow[dr, no head] & & P_2 \arrow[dl, no head] & \cong & \{1\} \arrow[dr, no head] & & \{2\} \arrow[dl, no head] \\
 & B & & & & \{1,2\} &
\end{tikzcd}
$$

Note: $B = P_1 \cap P_2$. As such, we may denote B as $P_{1,2}$.

We may choose our Sylow p-subgroup to be

$$U = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

We may choose our Sylow p-subgroup to be

$$U = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

From this, we obtain

$$B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

We may choose our Sylow p-subgroup to be

$$U = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

From this, we obtain

$$B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\} \quad P_1 = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\} \quad P_2 = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

We now recover the Weyl group.

We now recover the Weyl group.

$$U = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

We now recover the Weyl group.

$$U = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

$$T = \left\{ \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \right\}$$

We now recover the Weyl group.

$$U = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

$$T = \left\{ \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \right\} \quad N_G(T) = \left\langle \begin{bmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix} \right\rangle$$

We now recover the Weyl group.

$$U = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \right\} \qquad B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

$$T = \left\{ \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \right\} \quad N_G(T) = \left\langle \begin{bmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix} \right\rangle$$

$$W(G) = N_G(T)/T = \left\langle \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle \cong D_6$$

$\square$

Let G be a rank 2 group of Lie type with maximal parabolic subgroups P_1 and P_2.

Let G be a rank 2 group of Lie type with maximal parabolic subgroups P_1 and P_2.

$$\mathcal{P} = G/P_1 = \{gP_1 \mid g \in G\}$$

Let G be a rank 2 group of Lie type with maximal parabolic subgroups P_1 and P_2.

$$\mathcal{P} = G/P_1 = \{gP_1 \mid g \in G\}$$
$$\mathcal{L} = G/P_2 = \{gP_2 \mid g \in G\}$$

Let G be a rank 2 group of Lie type with maximal parabolic subgroups P_1 and P_2.

$$\mathcal{P} = G/P_1 = \{gP_1 \mid g \in G\}$$
$$\mathcal{L} = G/P_2 = \{gP_2 \mid g \in G\}$$

Now define the incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$

$$(xP_1, yP_2) \in \mathcal{I} \iff xP_1 \cap yP_2 \neq \emptyset$$

Let G be a rank 2 group of Lie type with maximal parabolic subgroups P_1 and P_2.

$$\mathcal{P} = G/P_1 = \{gP_1 \mid g \in G\}$$
$$\mathcal{L} = G/P_2 = \{gP_2 \mid g \in G\}$$

Now define the incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$

$$(xP_1, yP_2) \in \mathcal{I} \iff xP_1 \cap yP_2 \neq \emptyset$$

We refer to $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ as a **(thick) rank 2 building**.

EXAMPLE. Recall the linear model for projective plane $PG(2, \mathbb{F})$:

EXAMPLE. Recall the linear model for projective plane $PG(2, \mathbb{F})$:

$$\begin{aligned} \text{points} &\quad \longleftrightarrow \quad \text{1-dimensional subspaces of } \mathbb{F}^3 \\ \text{lines} &\quad \longleftrightarrow \quad \text{2-dimensional subspaces of } \mathbb{F}^3 \\ \text{incidence} &\quad \longleftrightarrow \quad \text{containment} \end{aligned}$$

EXAMPLE. Recall the linear model for projective plane $PG(2, \mathbb{F})$:

$$\begin{array}{rcl} \text{points} & \longleftrightarrow & \text{1-dimensional subspaces of } \mathbb{F}^3 \\ \text{lines} & \longleftrightarrow & \text{2-dimensional subspaces of } \mathbb{F}^3 \\ \text{incidence} & \longleftrightarrow & \text{containment} \end{array}$$

$$P_1 = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\}$$

EXAMPLE. Recall the linear model for projective plane $PG(2, \mathbb{F})$:

$$\begin{aligned} \text{points} \quad &\longleftrightarrow \quad \text{1-dimensional subspaces of } \mathbb{F}^3 \\ \text{lines} \quad &\longleftrightarrow \quad \text{2-dimensional subspaces of } \mathbb{F}^3 \\ \text{incidence} \quad &\longleftrightarrow \quad \text{containment} \end{aligned}$$

$$P_1 = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\} = \text{stabilizer of the point } \langle e_1 \rangle$$

EXAMPLE. Recall the linear model for projective plane $PG(2,\mathbb{F})$:

$$\begin{array}{rcl} \text{points} & \longleftrightarrow & \text{1-dimensional subspaces of } \mathbb{F}^3 \\ \text{lines} & \longleftrightarrow & \text{2-dimensional subspaces of } \mathbb{F}^3 \\ \text{incidence} & \longleftrightarrow & \text{containment} \end{array}$$

$$P_1 = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\} = \text{stabilizer of the point } \langle e_1 \rangle$$

$$P_2 = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$$

EXAMPLE. Recall the linear model for projective plane $PG(2, \mathbb{F})$:

$$\begin{array}{rcl} \text{points} & \longleftrightarrow & \text{1-dimensional subspaces of } \mathbb{F}^3 \\ \text{lines} & \longleftrightarrow & \text{2-dimensional subspaces of } \mathbb{F}^3 \\ \text{incidence} & \longleftrightarrow & \text{containment} \end{array}$$

$$P_1 = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\} = \text{stabilizer of the point } \langle e_1 \rangle$$

$$P_2 = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\} = \text{stabilizer of the line } \langle e_1, e_2 \rangle$$

EXAMPLE. Recall the linear model for projective plane $PG(2, \mathbb{F})$:

$$\begin{array}{rcl} \text{points} & \longleftrightarrow & \text{1-dimensional subspaces of } \mathbb{F}^3 \\ \text{lines} & \longleftrightarrow & \text{2-dimensional subspaces of } \mathbb{F}^3 \\ \text{incidence} & \longleftrightarrow & \text{containment} \end{array}$$

$$P_1 = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\} = \text{stabilizer of the point } \langle e_1 \rangle$$

$$P_2 = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\} = \text{stabilizer of the line } \langle e_1, e_2 \rangle$$

Thus G/P_1 and G/P_2 are the point set and line set of $PG(2, \mathbb{F})$. We conclude that buildings of type A_2 are nothing more than Desarguesian projective planes. □

Let G be a group of Lie type with fixed Borel subgroup B, fixed maximal torus $T < B$, maximal parabolics P_i, Weyl group W, and Lie algebra $\mathfrak{L}$.

Let G be a group of Lie type with fixed Borel subgroup B, fixed maximal torus $T < B$, maximal parabolics P_i, Weyl group W, and Lie algebra $\mathfrak{L}$.

$$G = BWB = \coprod_{w \in W} BwB$$ (**Bruhat decomposition**)

Let G be a group of Lie type with fixed Borel subgroup B, fixed maximal torus $T < B$, maximal parabolics P_i, Weyl group W, and Lie algebra $\mathfrak{L}$.

$$G = BWB = \coprod_{w \in W} BwB \quad (\textbf{Bruhat decomposition})$$

This allows the action of G on the coset spaces G/P_i to be formulated in terms of a composite action of W and B on these spaces.

Let G be a group of Lie type with fixed Borel subgroup B, fixed maximal torus $T < B$, maximal parabolics P_i, Weyl group W, and Lie algebra $\mathfrak{L}$.

$$G = BWB = \coprod_{w \in W} BwB \quad (\textbf{Bruhat decomposition})$$

This allows the action of G on the coset spaces G/P_i to be formulated in terms of a composite action of W and B on these spaces.

Each Borel orbit on G/P_i (**Schubert cell**) contains a unique T-invariant coset gP_i which may be identified with the coset $\alpha = wW_i \in W/W_i$ where $g \in BwB$ and $P_i = BW_iB$. Moreover, every $\alpha \in W/W_i$ is so realized. We denote this orbit by $\mathcal{B}_\alpha$.

Let G be a group of Lie type with fixed Borel subgroup B, fixed maximal torus $T < B$, maximal parabolics P_i, Weyl group W, and Lie algebra $\mathfrak{L}$.

$$\boldsymbol{G = BWB = \coprod_{w \in W} BwB} \quad (\textbf{Bruhat decomposition})$$

This allows the action of G on the coset spaces G/P_i to be formulated in terms of a composite action of W and B on these spaces.

Each Borel orbit on G/P_i (**Schubert cell**) contains a unique T-invariant coset gP_i which may be identified with the coset $\alpha = wW_i \in W/W_i$ where $g \in BwB$ and $P_i = BW_iB$. Moreover, every $\alpha \in W/W_i$ is so realized. We denote this orbit by $\mathcal{B}_\alpha$.

$$\text{Borel orbits } \mathcal{B}_\alpha \text{ of } G/P_i \xleftrightarrow{\textit{one-to-one}} \text{cosets } \alpha \in W/W_i$$

The action of W on W/W_i is equivalent to the contragredient action of W on the dual root space $\Phi^* \subset \mathfrak{H}$ given by

$$\boldsymbol{w(r^*) = (w^{-1}(r))^*}$$

The action of W on W/W_i is equivalent to the contragredient action of W on the dual root space $\Phi^* \subset \mathfrak{H}$ given by

$$\boldsymbol{w(r^*) = (w^{-1}(r))^*}$$

Hence we obtain embeddings $\boldsymbol{W/W_i \hookrightarrow \mathfrak{H} \subset \mathfrak{L}^U}$.

The action of W on W/W_i is equivalent to the contragredient action of W on the dual root space $\Phi^* \subset \mathfrak{H}$ given by

$$\boldsymbol{w(r^*) = (w^{-1}(r))^*}$$

Hence we obtain embeddings $\boldsymbol{W/W_i \hookrightarrow \mathfrak{H} \subset \mathfrak{L}^U}$.

But this means we now have an embedded transversal for the Borel orbits $\mathcal{B}_\alpha$.

The action of W on W/W_i is equivalent to the contragredient action of W on the dual root space $\Phi^* \subset \mathfrak{H}$ given by

$$\boldsymbol{w(r^*) = (w^{-1}(r))^*}$$

Hence we obtain embeddings $\boldsymbol{W/W_i \hookrightarrow \mathfrak{H} \subset \mathfrak{L}^U}$.

But this means we now have an embedded transversal for the Borel orbits $\mathcal{B}_\alpha$.

Every object in $\mathcal{B}_\alpha$ will embed in $\alpha \oplus \mathfrak{L}^+$, however very few of the vectors in $\alpha \oplus \mathfrak{L}^+$ will correspond to such embedded objects.

The action of W on W/W_i is equivalent to the contragredient action of W on the dual root space $\Phi^* \subset \mathfrak{H}$ given by

$$\boldsymbol{w(r^*) = (w^{-1}(r))^*}$$

Hence we obtain embeddings $\boldsymbol{W/W_i \hookrightarrow \mathfrak{H} \subset \mathfrak{L}^U}$.

But this means we now have an embedded transversal for the Borel orbits $\mathcal{B}_\alpha$.

Every object in $\mathcal{B}_\alpha$ will embed in $\alpha \oplus \mathfrak{L}^+$, however very few of the vectors in $\alpha \oplus \mathfrak{L}^+$ will correspond to such embedded objects.

Hence we need some way of identifying which vectors in $\alpha \oplus \mathfrak{L}^+$ represent embedded objects from $\mathcal{B}_\alpha$.

For each $\alpha \in W/W_i$ we define the set

$$\boldsymbol{\alpha}^{\mathbf{neg}} = \{\boldsymbol{r} \in \boldsymbol{\Phi}^+ \mid \boldsymbol{\alpha}(\boldsymbol{r}) < \mathbf{0}\}$$

For each $\alpha \in W/W_i$ we define the set

$$\boldsymbol{\alpha}^{\mathbf{neg}} = \{\boldsymbol{r} \in \boldsymbol{\Phi}^+ \mid \boldsymbol{\alpha}(\boldsymbol{r}) < \mathbf{0}\}$$

We now define the **expanse** of α to be

$$\boldsymbol{\mathfrak{L}}^+(\boldsymbol{\alpha}) = \left\{\sum \boldsymbol{\lambda}_{\boldsymbol{r}} \boldsymbol{e}_{\boldsymbol{r}} \in \boldsymbol{\mathfrak{L}}^+ \mid \boldsymbol{r} \in \boldsymbol{\alpha}^{\mathbf{neg}}\right\}$$

For each $\alpha \in W/W_i$ we define the set

$$\boldsymbol{\alpha}^{\mathbf{neg}} = \{\boldsymbol{r} \in \boldsymbol{\Phi}^+ \mid \boldsymbol{\alpha}(\boldsymbol{r}) < \mathbf{0}\}$$

We now define the **expanse** of α to be

$$\boldsymbol{\mathfrak{L}}^+(\boldsymbol{\alpha}) = \left\{\sum \boldsymbol{\lambda}_r \boldsymbol{e}_r \in \boldsymbol{\mathfrak{L}}^+ \mid \boldsymbol{r} \in \boldsymbol{\alpha}^{\mathbf{neg}}\right\}$$

Then

$$\boldsymbol{\mathcal{B}}_{\alpha} = \boldsymbol{\alpha} \oplus \boldsymbol{\mathfrak{L}}^+(\boldsymbol{\alpha})$$

For each $\alpha \in W/W_i$ we define the set

$$\boldsymbol{\alpha}^{\mathbf{neg}} = \{\boldsymbol{r} \in \boldsymbol{\Phi}^+ \mid \boldsymbol{\alpha}(\boldsymbol{r}) < \mathbf{0}\}$$

We now define the **expanse** of α to be

$$\mathfrak{L}^+(\boldsymbol{\alpha}) = \left\{\sum \boldsymbol{\lambda_r e_r} \in \mathfrak{L}^+ \mid \boldsymbol{r} \in \boldsymbol{\alpha}^{\mathbf{neg}}\right\}$$

Then

$$\boldsymbol{\mathcal{B}_\alpha} = \boldsymbol{\alpha} \oplus \mathfrak{L}^+(\boldsymbol{\alpha})$$

Note: This is a group-free description of the objects in each Borel orbit $\mathcal{B}_\alpha$. Thus a full determination of the objects in the embedded building depends only on the action of the Weyl group.

Incidence: $\alpha + \mathfrak{a} \in \mathcal{B}_\alpha$ is incident to $\beta + \mathfrak{b} \in \mathcal{B}_\beta$ if and only if

Incidence: $\alpha + \mathfrak{a} \in \mathcal{B}_\alpha$ is incident to $\beta + \mathfrak{b} \in \mathcal{B}_\beta$ if and only if

1. $\alpha(r)\beta(r) \geq 0$ for all $r \in \Phi^+$ (**Weyl incidence**)

Incidence: $\alpha + \mathfrak{a} \in \mathcal{B}_\alpha$ is incident to $\beta + \mathfrak{b} \in \mathcal{B}_\beta$ if and only if

1. $\alpha(r)\beta(r) \geq 0$ for all $r \in \Phi^+$ (**Weyl incidence**)
2. the projection of $[\alpha + \mathfrak{a}, \beta + \mathfrak{b}]$ onto $\mathfrak{L}^+(\alpha) \cap \mathfrak{L}^+(\beta)$ is zero

Incidence: $\alpha + \mathfrak{a} \in \mathcal{B}_\alpha$ is incident to $\beta + \mathfrak{b} \in \mathcal{B}_\beta$ if and only if

1. $\alpha(r)\beta(r) \geq 0$ for all $r \in \Phi^+$ (**Weyl incidence**)
2. the projection of $[\alpha + \mathfrak{a}, \beta + \mathfrak{b}]$ onto $\mathfrak{L}^+(\alpha) \cap \mathfrak{L}^+(\beta)$ is zero

Note: This coincides with the previously defined incidence on the pre-embedded objects of the geometry (nonempty intersection of cosets).

EXAMPLE. We illustrate the embedding procedure for the classical projective plane $PG(2, q)$.

EXAMPLE. We illustrate the embedding procedure for the classical projective plane $PG(2, q)$.

	α	$\mathfrak{a}$	$\lvert\mathcal{B}_\alpha\rvert$
$\mathcal{P}$	r_1^*	0	1
	$-r_1^* + r_2^*$	$\lambda_{r_1} e_{r_1}$	q
	$-r_2^*$	$\lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2}$	q^2
$\mathcal{L}$	r_2^*	0	1
	$r_1^* - r_2^*$	$\lambda_{r_2} e_{r_2}$	q
	$-r_1^*$	$\lambda_{r_1} e_{r_1} + \lambda_{r_1+r_2} e_{r_1+r_2}$	q^2

Table: *Objects of the embedded building of type A_2 in $\mathfrak{L}^U = \mathfrak{H} \oplus \mathfrak{L}^+$*
Each embedded object is of the form $\alpha + \mathfrak{a}$ for $\alpha \in \mathfrak{H}$ and $\mathfrak{a} \in \mathfrak{L}^+(\alpha)$

		lines	
	r_2^*	$r_1^* - r_2^* + \gamma_{r_2} e_{r_2}$	$-r_1^* + \gamma_{r_1} e_{r_1} + \gamma_{r_1+r_2} e_{r_1+r_2}$
points r_1^*	1	1	0
points $-r_1^* + r_2^* + \lambda_{r_1} e_{r_1}$	1	0	δ_{ab}
points $-r_2^* + \lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2}$	0	δ_{cd}	δ_{ef}

Table: *Incidence in the embedded building of type A_2.*

Each of δ_{ab}, δ_{cd} and δ_{ef} is the kronecker delta function, where $a = 2\lambda_{r_1}$, $b = 3\gamma_{r_1}$; $c = 2\lambda_{r_2}$, $d = 3\gamma_{r_2}$; and $e = \lambda_{r_1+r_2}$, $f = \gamma_{r_1+r_2} + \lambda_{r_1}\gamma_{r_2}$.

We perform the computation for incidence between points and lines in the largest Borel orbits. These orbits are $\mathcal{B}_{-r_2^*}$ and $\mathcal{B}_{-r_1^*}$ respectively.

We perform the computation for incidence between points and lines in the largest Borel orbits. These orbits are $\mathcal{B}_{-r_2^*}$ and $\mathcal{B}_{-r_1^*}$ respectively.

Both orbits have size q^2 and generate the classical **biaffine plane**, i.e., the classical affine plane with one parallel class removed.

We perform the computation for incidence between points and lines in the largest Borel orbits. These orbits are $\mathcal{B}_{-r_2^*}$ and $\mathcal{B}_{-r_1^*}$ respectively.

Both orbits have size q^2 and generate the classical **biaffine plane**, i.e., the classical affine plane with one parallel class removed.

Convention: We denote the scalar multiple of the root vector $e_{ir_1+jr_2}$ by p_{ij} for each point (p) and by ℓ_{ij} for each line $[\ell]$.

We perform the computation for incidence between points and lines in the largest Borel orbits. These orbits are $\mathcal{B}_{-r_2^*}$ and $\mathcal{B}_{-r_1^*}$ respectively.

Both orbits have size q^2 and generate the classical **biaffine plane**, i.e., the classical affine plane with one parallel class removed.

Convention: We denote the scalar multiple of the root vector $e_{ir_1+jr_2}$ by p_{ij} for each point (p) and by ℓ_{ij} for each line $[\ell]$.

$$(p) = -r_2^* + p_{01}e_{r_2} + p_{11}e_{r_1+r_2} \in \mathcal{B}_{-r_2^*}$$

We perform the computation for incidence between points and lines in the largest Borel orbits. These orbits are $\mathcal{B}_{-r_2^*}$ and $\mathcal{B}_{-r_1^*}$ respectively.

Both orbits have size q^2 and generate the classical **biaffine plane**, i.e., the classical affine plane with one parallel class removed.

Convention: We denote the scalar multiple of the root vector $e_{ir_1+jr_2}$ by p_{ij} for each point (p) and by ℓ_{ij} for each line $[\ell]$.

$$(p) = -r_2^* + p_{01}e_{r_2} + p_{11}e_{r_1+r_2} \in \mathcal{B}_{-r_2^*}$$

$$[\ell] = -r_1^* + \ell_{10}e_{r_1} + \ell_{11}e_{r_1+r_2} \in \mathcal{B}_{-r_1^*}$$

As $(-r_2^*)(r)(-r_1^*)(r) \geq 0$ for all $r \in \Phi^+$ we have Weyl incidence.

As $(-r_2^*)(r)(-r_1^*)(r) \geq 0$ for all $r \in \Phi^+$ we have Weyl incidence.

Thus point (p) is incident to line $[\ell]$ precisely when the projection of $[(p), [\ell]]$ onto $\mathfrak{L}^+(-r_2^*) \cap \mathfrak{L}^+(-r_1^*)$ is zero.

As $(-r_2^*)(r)(-r_1^*)(r) \geq 0$ for all $r \in \Phi^+$ we have Weyl incidence.

Thus point (p) is incident to line $[\ell]$ precisely when the projection of $[(p), [\ell]]$ onto $\mathfrak{L}^+(-r_2^*) \cap \mathfrak{L}^+(-r_1^*)$ is zero.

$$\begin{aligned}
[(p), [\ell]] &= [-r_2^*, -r_1^*] + [p_{01}e_{r_2}, -r_1^*] + [p_{11}e_{r_1+r_2}, -r_1^*] + [-r_2^*, \ell_{10}e_{r_1}] + \\
&\quad [p_{01}e_{r_2}, \ell_{10}e_{r_1}] + [p_{11}e_{r_1+r_2}, \ell_{10}e_{r_1}] + [-r_2^*, \ell_{11}e_{r_1+r_2}] + \\
&\quad [p_{01}e_{r_2}, \ell_{11}e_{r_1+r_2}] + [p_{11}e_{r_1+r_2}, \ell_{11}e_{r_1+r_2}] \\
&= (p_{11} - p_{01}\ell_{10} - \ell_{11})e_{r_1+r_2}
\end{aligned}$$

As $(-r_2^*)(r)(-r_1^*)(r) \geq 0$ for all $r \in \Phi^+$ we have Weyl incidence.

Thus point (p) is incident to line $[\ell]$ precisely when the projection of $[(p),[\ell]]$ onto $\mathfrak{L}^+(-r_2^*) \cap \mathfrak{L}^+(-r_1^*)$ is zero.

$$\begin{aligned}
[(p),[\ell]] &= [-r_2^*,-r_1^*] + [p_{01}e_{r_2},-r_1^*] + [p_{11}e_{r_1+r_2},-r_1^*] + [-r_2^*,\ell_{10}e_{r_1}] + \\
&\quad [p_{01}e_{r_2},\ell_{10}e_{r_1}] + [p_{11}e_{r_1+r_2},\ell_{10}e_{r_1}] + [-r_2^*,\ell_{11}e_{r_1+r_2}] + \\
&\quad [p_{01}e_{r_2},\ell_{11}e_{r_1+r_2}] + [p_{11}e_{r_1+r_2},\ell_{11}e_{r_1+r_2}] \\
&= (p_{11} - p_{01}\ell_{10} - \ell_{11})e_{r_1+r_2}
\end{aligned}$$

As $\mathfrak{L}^+(-r_2^*) \cap \mathfrak{L}^+(-r_1^*) = \langle e_{r_1+r_2} \rangle$ we conclude that (p) and $[\ell]$ are incident if and only if $p_{11} - p_{01}\ell_{10} - \ell_{11} = 0$,

As $(-r_2^*)(r)(-r_1^*)(r) \geq 0$ for all $r \in \Phi^+$ we have Weyl incidence.

Thus point (p) is incident to line $[\ell]$ precisely when the projection of $[(p),[\ell]]$ onto $\mathfrak{L}^+(-r_2^*) \cap \mathfrak{L}^+(-r_1^*)$ is zero.

$$\begin{aligned}
[(p),[\ell]] &= [-r_2^*, -r_1^*] + [p_{01}e_{r_2}, -r_1^*] + [p_{11}e_{r_1+r_2}, -r_1^*] + [-r_2^*, \ell_{10}e_{r_1}] + \\
&\quad [p_{01}e_{r_2}, \ell_{10}e_{r_1}] + [p_{11}e_{r_1+r_2}, \ell_{10}e_{r_1}] + [-r_2^*, \ell_{11}e_{r_1+r_2}] + \\
&\quad [p_{01}e_{r_2}, \ell_{11}e_{r_1+r_2}] + [p_{11}e_{r_1+r_2}, \ell_{11}e_{r_1+r_2}] \\
&= (p_{11} - p_{01}\ell_{10} - \ell_{11})e_{r_1+r_2}
\end{aligned}$$

As $\mathfrak{L}^+(-r_2^*) \cap \mathfrak{L}^+(-r_1^*) = \langle e_{r_1+r_2} \rangle$ we conclude that (p) and $[\ell]$ are incident if and only if $p_{11} - p_{01}\ell_{10} - \ell_{11} = 0$,

i.e., $\boldsymbol{p_{11} - \ell_{11} = p_{01}\ell_{10}}$

EXAMPLE. We provide objects of the embedded generalized quadrangle of type B_2.

EXAMPLE. We provide objects of the embedded generalized quadrangle of type B_2.

	α	$\mathfrak{a}$	$\lvert\mathcal{B}_\alpha\rvert$
$\mathcal{P}$	r_1^*	0	1
	$-r_1^* + 2r_2^*$	$\lambda_{r_1} e_{r_1}$	q
	$r_1^* - 2r_2^*$	$\lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2}$	q^2
	$-r_1^*$	$\lambda_{r_1} e_{r_1} + \lambda_{r_1+r_2} e_{r_1+r_2} + \lambda_{2r_1+r_2} e_{2r_1+r_2}$	q^3
$\mathcal{L}$	r_2^*	0	1
	$r_1^* - r_2^*$	$\lambda_{r_2} e_{r_2}$	q
	$-r_1^* + r_2^*$	$\lambda_{r_1} e_{r_1} + \lambda_{2r_1+r_2} e_{2r_1+r_2}$	q^2
	$-r_2^*$	$\lambda_{r_2} e_{r_2} + \lambda_{r_1+r_2} e_{r_1+r_2} + \lambda_{2r_1+r_2} e_{2r_1+r_2}$	q^3

Table: *Objects in the embedded building of type B_2*

Our interest in the rank 2 case stems from the fact that rank 2 buildings are examples of **generalized polygons**.

Our interest in the rank 2 case stems from the fact that rank 2 buildings are examples of **generalized polygons**.

group	polygon

Our interest in the rank 2 case stems from the fact that rank 2 buildings are examples of **generalized polygons**.

group	polygon
$A_2(q)$	triangle

Our interest in the rank 2 case stems from the fact that rank 2 buildings are examples of **generalized polygons**.

group	polygon
$A_2(q)$	triangle
$B_2(q), C_2(q)$	quadrangle
${}^2A_3(q)$	quadrangle
${}^2A_4(q)$	quadrangle

Our interest in the rank 2 case stems from the fact that rank 2 buildings are examples of **generalized polygons**.

group	polygon
$A_2(q)$	triangle
$B_2(q), C_2(q)$	quadrangle
${}^2A_3(q)$	quadrangle
${}^2A_4(q)$	quadrangle
$G_2(q)$	hexagon
${}^3D_4(q)$	hexagon

Our interest in the rank 2 case stems from the fact that rank 2 buildings are examples of **generalized polygons**.

group	polygon
$A_2(q)$	triangle
$B_2(q), C_2(q)$	quadrangle
${}^2A_3(q)$	quadrangle
${}^2A_4(q)$	quadrangle
$G_2(q)$	hexagon
${}^3D_4(q)$	hexagon
${}^2F_4(2^{2m+1})$	octagon

In our attempt to construct families whose behavior would resemble that of the balanced generalized polygons, we felt that Dynkin diagrams would provide the most promising pathway.

Objects of type $\widetilde{A}_1$

the key to everything

In our attempt to construct families whose behavior would resemble that of the balanced generalized polygons, we felt that Dynkin diagrams would provide the most promising pathway.

However, the supply is already exhausted:

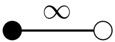
A_2		generalized triangle
B_2		generalized quadrangle
G_2		generalized hexagon

Objects of type $\widetilde{A}_1$

the key to everything

Or is it?

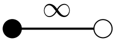
$\widetilde{A}_1$



Objects of type $\widetilde{A}_1$

the key to everything

Or is it?

$\widetilde{A}_1$ 

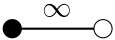
$\widetilde{A}_1$ is an **extended Dynkin diagram** obtained by adjoining an imaginary root to the Dynkin diagram of type A_1.

Objects of type $\widetilde{A}_1$

the key to everything

Or is it?

$\widetilde{A}_1$



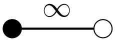
$\widetilde{A}_1$ is an **extended Dynkin diagram** obtained by adjoining an imaginary root to the Dynkin diagram of type A_1.

The Weyl group $W(\widetilde{A}_1)$ is the infinite dihedral group D_∞.

Objects of type $\widetilde{A}_1$

the key to everything

Or is it?

$\widetilde{A}_1$ 

$\widetilde{A}_1$ is an **extended Dynkin diagram** obtained by adjoining an imaginary root to the Dynkin diagram of type A_1.

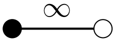
The Weyl group $W(\widetilde{A}_1)$ is the infinite dihedral group D_∞.

The affine root system $\Phi(\widetilde{A}_1)$ is infinite.

Objects of type $\widetilde{A}_1$

the key to everything

Or is it?

$\widetilde{A}_1$ 

$\widetilde{A}_1$ is an **extended Dynkin diagram** obtained by adjoining an imaginary root to the Dynkin diagram of type A_1.

The Weyl group $W(\widetilde{A}_1)$ is the infinite dihedral group D_∞.

The affine root system $\Phi(\widetilde{A}_1)$ is infinite.

Hence, the affine Lie algebra $\mathfrak{L}(\widetilde{A}_1)$ is infinite-dimensional (although its Cartan subalgebra is 2-dimensional).

In fact, there are two nonisomorphic affine Lie algebras of type $\widetilde{A}_1$, with respective Cartan matrices:

In fact, there are two nonisomorphic affine Lie algebras of type $\widetilde{A}_1$, with respective Cartan matrices:

$$M_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

In fact, there are two nonisomorphic affine Lie algebras of type $\widetilde{A}_1$, with respective Cartan matrices:

$$M_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

We decided to work with M_1.

In fact, there are two nonisomorphic affine Lie algebras of type $\widetilde{A}_1$, with respective Cartan matrices:

$$M_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

We decided to work with M_1.

From here, we generate the set Φ^+ of positive roots:

$$r_1,\ r_2,\ r_1 + r_2,\ 2r_1 + r_2,\ r_1 + 2r_2,\ 2r_1 + 2r_2,\ \ldots\ldots,$$
$$ir_1 + (i-1)r_2,\ (i-1)r_1 + ir_2,\ ir_1 + ir_2, \ldots\ldots$$

Note: For each $|i| \geq 2$, the root space $\mathfrak{L}_r$ where $r = ir_1 + ir_2$ is isotropic with respect to the Killing form, so is 2-dimensional.

Note: For each $|i| \geq 2$, the root space $\mathfrak{L}_r$ where $r = ir_1 + ir_2$ is isotropic with respect to the Killing form, so is 2-dimensional.

In this case, we write $\mathfrak{L}_r = \langle e_{ir_1+ir_2}, e'_{ir_1+ir_2} \rangle$.

Note: For each $|i| \geq 2$, the root space $\mathfrak{L}_r$ where $r = ir_1 + ir_2$ is isotropic with respect to the Killing form, so is 2-dimensional.

In this case, we write $\mathfrak{L}_r = \langle e_{ir_1+ir_2}, e'_{ir_1+ir_2} \rangle$.

This cannot occur in the (finite-dimensional) Lie algebras of groups of Lie type, where all root spaces $\mathfrak{L}_r$ are 1-dimensional.

Note: For each $|i| \geq 2$, the root space $\mathfrak{L}_r$ where $r = ir_1 + ir_2$ is isotropic with respect to the Killing form, so is 2-dimensional.

In this case, we write $\mathfrak{L}_r = \langle e_{ir_1+ir_2}, e'_{ir_1+ir_2} \rangle$.

This cannot occur in the (finite-dimensional) Lie algebras of groups of Lie type, where all root spaces $\mathfrak{L}_r$ are 1-dimensional.

Personally, I believe this to be a plausible explanation as to why there do not exist generalized polygons of arbitrary even girth.

Observe that $\Phi^+(\widetilde{A}_1)$ contains $\Phi^+(A_2)$ and $\Phi^+(B_2)$ as initial segments:

Observe that $\Phi^+(\widetilde{A}_1)$ contains $\Phi^+(A_2)$ and $\Phi^+(B_2)$ as initial segments:

$$\boldsymbol{\Phi^+(\widetilde{A}_1) =}$$
$$\{r_1,\ r_2,\ r_1+r_2,\ 2r_1+r_2,\ r_1+2r_2,\ 2r_1+2r_2,\ \ldots\ldots\}$$

Observe that $\Phi^+(\widetilde{A}_1)$ contains $\Phi^+(A_2)$ and $\Phi^+(B_2)$ as initial segments:

$$\boldsymbol{\Phi^+(\widetilde{A}_1) =}$$
$$\{r_1,\ r_2,\ r_1 + r_2,\ 2r_1 + r_2,\ r_1 + 2r_2,\ 2r_1 + 2r_2,\ \ldots\ldots\}$$

$$\boldsymbol{\Phi^+(A_2) =}$$
$$\boldsymbol{\{r_1,\ r_2,\ r_1 + r_2\}},\ 2r_1 + r_2,\ r_1 + 2r_2,\ 2r_1 + 2r_2,\ \ldots\ldots$$

Observe that $\Phi^+(\widetilde{A}_1)$ contains $\Phi^+(A_2)$ and $\Phi^+(B_2)$ as initial segments:

$$\boldsymbol{\Phi^+(\widetilde{A}_1) =}$$
$$\{r_1,\ r_2,\ r_1+r_2,\ 2r_1+r_2,\ r_1+2r_2,\ 2r_1+2r_2,\ \ldots\ldots\}$$

$$\boldsymbol{\Phi^+(A_2) =}$$
$$\boldsymbol{\{r_1,\ r_2,\ r_1+r_2\}},\ 2r_1+r_2,\ r_1+2r_2,\ 2r_1+2r_2,\ \ldots\ldots$$

$$\boldsymbol{\Phi^+(B_2) =}$$
$$\boldsymbol{\{r_1,\ r_2,\ r_1+r_2,\ 2r_1+r_2\}},\ r_1+2r_2,\ 2r_1+2r_2,\ \ldots\ldots$$

Observe that $\Phi^+(\widetilde{A}_1)$ contains $\Phi^+(A_2)$ and $\Phi^+(B_2)$ as initial segments:

$$\boldsymbol{\Phi^+(\widetilde{A}_1) =}$$
$$\{r_1,\ r_2,\ r_1+r_2,\ 2r_1+r_2,\ r_1+2r_2,\ 2r_1+2r_2,\ \ldots\ldots\}$$

$$\boldsymbol{\Phi^+(A_2) =}$$
$$\boldsymbol{\{r_1,\ r_2,\ r_1+r_2\}},\ 2r_1+r_2,\ r_1+2r_2,\ 2r_1+2r_2,\ \ldots\ldots$$

$$\boldsymbol{\Phi^+(B_2) =}$$
$$\boldsymbol{\{r_1,\ r_2,\ r_1+r_2,\ 2r_1+r_2\}},\ r_1+2r_2,\ 2r_1+2r_2,\ \ldots\ldots$$

So, what happens if we truncate $\Phi^+(\widetilde{A}_1)$ at increasingly larger initial segments?

Since we obtain the same asymptotics working with the largest Borel orbits of points and lines, we may restrict our truncated geometries to their affine parts:

Since we obtain the same asymptotics working with the largest Borel orbits of points and lines, we may restrict our truncated geometries to their affine parts:

$$\textbf{points}: \ \boldsymbol{\mathcal{B}_{-r_2^*} = -r_2^* \oplus \mathfrak{L}^+(-r_2^*)}$$

Since we obtain the same asymptotics working with the largest Borel orbits of points and lines, we may restrict our truncated geometries to their affine parts:

$$\begin{aligned} \textbf{points}: \ \boldsymbol{\mathcal{B}_{-r_2^*} = -r_2^* \oplus \mathfrak{L}^+(-r_2^*)} \\ \textbf{lines}: \ \boldsymbol{\mathcal{B}_{-r_1^*} = -r_1^* \oplus \mathfrak{L}^+(-r_1^*)} \end{aligned}$$

Since we obtain the same asymptotics working with the largest Borel orbits of points and lines, we may restrict our truncated geometries to their affine parts:

$$\begin{aligned} \textbf{points}: \ \mathcal{B}_{-r_2^*} &= -r_2^* \oplus \mathfrak{L}^+(-r_2^*) \\ \textbf{lines}: \ \mathcal{B}_{-r_1^*} &= -r_1^* \oplus \mathfrak{L}^+(-r_1^*) \end{aligned}$$

This eliminates dependence on the Weyl group.

Since we obtain the same asymptotics working with the largest Borel orbits of points and lines, we may restrict our truncated geometries to their affine parts:

$$\begin{aligned} \textbf{points}: \ & \boldsymbol{\mathcal{B}_{-r_2^*} = -r_2^* \oplus \mathfrak{L}^+(-r_2^*)} \\ \textbf{lines}: \ & \boldsymbol{\mathcal{B}_{-r_1^*} = -r_1^* \oplus \mathfrak{L}^+(-r_1^*)} \end{aligned}$$

This eliminates dependence on the Weyl group.

Since the notion of expanse has already eliminated dependence on the Borel subgroup, our affine truncated geometries now have a completely group-free formulation.

$CD(k,q)$

Graphs $CD(k,q)$ arise as connected components of incidence graphs of affine parts of truncated buildings of type $\widetilde{A}_1$.

$CD(k, q)$

Graphs $CD(k, q)$ arise as connected components of incidence graphs of affine parts of truncated buildings of type $\widetilde{A}_1$.

Truncating after the initial three positive root vectors gives the classical biaffine plane. After the initial four positive root vectors it gives the affine part of the generalized quadrangle of type B_2.

Graphs $CD(k,q)$ arise as connected components of incidence graphs of affine parts of truncated buildings of type $\widetilde{A}_1$.

Truncating after the initial three positive root vectors gives the classical biaffine plane. After the initial four positive root vectors it gives the affine part of the generalized quadrangle of type B_2.

The affine part of the generalized hexagon does not appear in our series (perhaps due to the root space $\mathfrak{L}_{2r_1+2r_2}$ being isotropic ??).

The End
Thank you!

Vasya Ustimenko, Ivar Stakgold, Me, Felix, Joe Hemmeter (seated)