A characterization of Q-polynomial distance-regular graphs using the intersection numbers

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Overview

Let Γ denote a distance-regular graph with diameter $D \geq 3$.
Assume Γ is primitive.
We use the intersection numbers of Γ to find a positive semidefinite matrix G with integer entries.
We show that G has determinant zero if and only if Γ is Q-polynomial.

History

Let Γ denote a distance-regular graph with diameter $D \geq 3$.
In the literature there are a number of characterizations for the Q-polynomial condition on Γ.

- There is the balanced set characterization (Terwilliger 1987, 1995).
- There is a characterization involving the dual distance matrices (Terwilliger 1995).
- There is a characterization involving the intersection numbers a_i (Pascasio 2008; cf. Hanson 2013).
- There is a characterization involving a tail in a representation diagram (Jurišić, Terwilliger, and Žitnik, 2010).
- There is a characterization involving a pair of primitive idempotents (Kurihara and Nozaki 2012; cf. Nomura and Terwilliger 2011).

Preliminaries

- $\Gamma = (X, R)$: a finite, undirected, connected graph, without loops or multiple edges
  - X : vertex set
  - R : edge set
- ∂ : the shortest path-length distance function for Γ.
- $D := \max\{\partial(x,y) | x, y \in X\}$: the *diameter* of Γ
- $\Gamma_i(x)$: the set of vertices at distance i from x

$$\Gamma_i(x) = \{y \in V | \partial(x,y) = i\}$$

- $\Gamma(x) = \Gamma_1(x)$
- The graph Γ is called distance-regular whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all $x, y \in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$$

  is independent of x and y.
- The integers p^h_{ij} are called the intersection numbers of Γ.

From now on we assume Γ is distance-regular with diameter $D \geq 3$. We abbreviate

$$\begin{aligned}
c_i &:= p^i_{1,i-1} && (1 \leq i \leq D),\\
a_i &:= p^i_{1i} && (1 \leq i \leq D),\\
b_i &:= p^i_{1,i+1} && (0 \leq i \leq D-1),\\
k_i &:= p^0_{ii} && (0 \leq i \leq D),\\
c_0 &= 0,\\
b_D &= 0.
\end{aligned}$$

Γ is regular with valency $k = b_0$ and $c_i + a_i + b_i = k$ $(0 \leq i \leq D)$.

By [BCN] (Brouwer, Cohen, and Neumaier 1989) the following holds for $0 \leq h, i, j \leq D$:

$$p^h_{ij} = \begin{cases} = 0 & \text{if one of } h, i, j \text{ is greater than the sum of the other two,} \\ \neq 0 & \text{if one of } h, i, j \text{ equals the sum of the other two.} \end{cases}$$

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For $0 \le i \le D$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if} \quad \partial(x, y) = i, \\ 0 & \text{if} \quad \partial(x, y) \neq i, \end{cases} \qquad x, y \in X.$$

We call A_i the i-th distance matrix of Γ.
We call $A = A_1$ the adjacency matrix of Γ.
Observe that A_i is real and symmetric for $0 \le i \le D$.
Note that $A_0 = I$, where I is the identity matrix.
Observe that

- $\sum_{i=0}^{D} A_i = J$, where J is the all-ones matrix in $\mathrm{Mat}_X(\mathbb{C})$.
- For $0 \le i, j \le D$, $A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h$.

Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A.
The matrices $A_0, A_1, ..., A_D$ form a basis for M.
We call M the Bose-Mesner algebra of Γ.
M has a basis $E_0, E_1, ..., E_D$ such that

$$E_0 = |X|^{-1}J;$$

$$\sum_{i=0}^{D} E_i = I;$$

$$E_i^t = E_i \qquad (0 \leq i \leq D);$$

$$\overline{E_i} = E_i \qquad (0 \leq i \leq D);$$

$$E_i E_j = \delta_{ij} E_i \qquad (0 \leq i, j \leq D).$$

The matrices $E_0, E_1, ..., E_D$ are called the primitive idempotents of Γ, and E_0 is called the trivial idempotent.
For $0 \leq i \leq D$ let m_i denote the rank of E_i.

Let λ denote an indeterminate.
Define polynomials $\{v_i\}_{i=0}^{D+1}$ in $\mathbb{C}[\lambda]$ by $v_0 = 1$, $v_1 = \lambda$, and

$$\lambda v_i = c_{i+1}v_{i+1} + a_i v_i + b_{i-1}v_{i-1} \qquad (1 \leq i \leq D),$$

where $c_{D+1} := 1$.
The following hold:

- $deg\ v_i = i\ (0 \leq i \leq D+1)$;
- the coefficient of λ^i in v_i is $(c_1 c_2 \cdots c_i)^{-1}$ $(0 \leq i \leq D+1)$;
- $v_i(A) = A_i\ (0 \leq i \leq D)$;
- $v_{D+1}(A) = 0$;
- the distinct eigenvalues of Γ are precisely the zeros of v_{D+1}; call these $\theta_0, \theta_1, ..., \theta_D$.

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Define polynomials $\{u_i\}_{i=0}^{D}$ in $\mathbb{C}[\lambda]$ by $u_0 = 1$, $u_1 = \lambda/k$, and

$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad (1 \leq i \leq D-1).$$

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$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad (1 \leq i \leq D-1).$$

Observe that $u_i = v_i/k_i\ (0 \leq i \leq D)$.

$$A_j = \sum_{i=0}^{D} v_j(\theta_i) E_i \qquad (0 \le j \le D), \tag{1}$$

$$E_j = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) A_i \qquad (0 \le j \le D). \tag{2}$$

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Definition 1.

Let $S \in \mathrm{Mat}_{D+1}(\mathbb{C})$ denote the transition matrix from the basis $\{A_i\}_{i=0}^{D}$ of M to the basis $\{E_i\}_{i=0}^{D}$ of M. Thus

$$E_j = \sum_{i=0}^{D} S_{ij} A_i, \qquad A_j = \sum_{i=0}^{D} (S^{-1})_{ij} E_i \qquad (0 \le j \le D).$$

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For $0 \le i, j \le D$,

$$S_{ij} = |X|^{-1} m_j u_i(\theta_j), \qquad (S^{-1})_{ij} = v_j(\theta_i).$$

Let $\circ$ denote the entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{C})$.
Note that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$. So M is closed under $\circ$.
There exist scalars $q^h_{ij} \in \mathbb{C}$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \qquad (0 \leq i, j \leq D). \tag{3}$$

We call the q^h_{ij} the Krein parameters of Γ.
These parameters are real and nonnegative for $0 \leq h, i, j \leq D$.

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These parameters are real and nonnegative for $0 \le h, i, j \le D$.
The graph Γ is said to be Q-polynomial with respect to the ordering $E_0, E_1, ..., E_D$ whenever the following hold for $0 \le h, i, j \le D$:

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There exist scalars $q_{ij}^h \in \mathbb{C}$ such that

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We call the q_{ij}^h the Krein parameters of Γ.
These parameters are real and nonnegative for $0 \leq h, i, j \leq D$.
The graph Γ is said to be Q-polynomial with respect to the ordering $E_0, E_1, ..., E_D$ whenever the following hold for $0 \leq h, i, j \leq D$:

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Let E denote a primitive idempotent of Γ.
We say that Γ is Q-polynomial with respect to E whenever there exists a Q-polynomial ordering $E_0, E_1, ..., E_D$ of the primitive idempotents such that $E = E_1$.

Fix a vertex $x \in X$.
For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if} \quad \partial(x, y) = i, \\ 0 & \text{if} \quad \partial(x, y) \neq i, \end{cases} \qquad y \in X.$$

We call E_i^* the i-th dual idempotent of Γ with respect to x.

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$$\begin{aligned} &\sum_{i=0}^{D} E_i^* = I; \\ &E_i^{*t} = E_i^* \qquad (0 \le i \le D); \\ &\overline{E_i^*} = E_i^* \qquad (0 \le i \le D); \\ &E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \le i, j \le D) \end{aligned}$$

By construction $E_0^*, E_1^*, ..., E_D^*$ are linearly independent.

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By construction $E_0^*, E_1^*, ..., E_D^*$ are linearly independent.
Let $M^* = M^*(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ with basis $E_0^*, E_1^*, ..., E_D^*$.
We call M^* the dual Bose-Mesner algebra of Γ with respect to x.

For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \qquad y \in X. \tag{4}$$

We call A_i^* the dual distance matrix of Γ with respect to x and E_i.

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We call A_i^* the dual distance matrix of Γ with respect to x and E_i. The matrices $A_0^*, A_1^*, ..., A_D^*$ form a basis for M^*.

$$A_0^* = I;$$

$$\sum_{i=0}^{D} A_i^* = |X|E_0^*;$$

$$A_i^{*t} = A_i^* \qquad (0 \leq i \leq D);$$

$$\overline{A_i^*} = A_i^* \qquad (0 \leq i \leq D);$$

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \qquad (0 \leq i, j \leq D).$$

From (1), (2) we have

$$A_j^* = m_j \sum_{i=0}^{D} u_i(\theta_j) E_i^* \qquad (0 \le j \le D), \tag{5}$$

$$E_j^* = |X|^{-1} \sum_{i=0}^{D} v_j(\theta_i) A_i^* \qquad (0 \le j \le D). \tag{6}$$

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The matrix $|X|S$ is the transition matrix from the basis $\{E_i^*\}_{i=0}^{D}$ of M^* to the basis $\{A_i^*\}_{i=0}^{D}$ of M^*.

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Thus

$$A_j^* = |X| \sum_{i=0}^{D} S_{ij} E_i^* \qquad (0 \le j \le D),$$

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The matrices $S^{alt}, (S^{-1})^{alt}, S'$

We now modify the matrices S, S^{-1} to obtain $D \times D$ matrices $S^{alt}, (S^{-1})^{alt}$ as follows:

$$(S^{alt})_{ij} = S_{ij} - S_{0j} \qquad (1 \le i, j \le D), \tag{7}$$

$$(S^{-1})^{alt}_{ij} = (S^{-1})_{ij} \qquad (1 \le i, j \le D). \tag{8}$$

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The matrices $S^{alt}, (S^{-1})^{alt}, S'$

Lemma 2.

The following (i)–(iv) hold.

(i) S^{alt} *is the transition matrix from* $\{A_2E_i^*A - AE_i^*A_2\}_{i=1}^D$ *to* $\{A_2A_i^*A - AA_i^*A_2\}_{i=1}^D$.

(ii) S^{alt} *is the transition matrix from* $\{A_3E_i^* - E_i^*A_3\}_{i=1}^D$ *to* $\{A_3A_i^* - A_i^*A_3\}_{i=1}^D$.

(iii) S^{alt} *is the transition matrix from* $\{A_2E_i^* - E_i^*A_2\}_{i=1}^D$ *to* $\{A_2A_i^* - A_i^*A_2\}_{i=1}^D$.

(iv) S^{alt} *is the transition matrix from* $\{AE_i^* - E_i^*A\}_{i=1}^D$ *to* $\{AA_i^* - A_i^*A\}_{i=1}^D$.

(v) $(S^{-1})^{alt}$ *and* S^{alt} *are inverses.*

Define a matrix

$$S' = \begin{bmatrix} S^{alt} & & & \mathbf{0} \\ & S^{alt} & & \\ & & S^{alt} & \\ \mathbf{0} & & & S^{alt} \end{bmatrix},$$

where S^{alt} is from (7).

Observe that S' is $4D \times 4D$.

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Lemma 3.

$det(S') = \left(det(S^{alt})\right)^4$. Moreover S' is invertible.

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Corollary 4.

The matrix S' is the transition matrix from
$\{A_2E_i^*A - AE_i^*A_2\}_{i=1}^D, \{A_3E_i^* - E_i^*A_3\}_{i=1}^D, \{A_2E_i^* - E_i^*A_2\}_{i=1}^D,$
$\{AE_i^* - E_i^*A\}_{i=1}^D$
to
$\{A_2A_i^*A - AA_i^*A_2\}_{i=1}^D, \{A_3A_i^* - A_i^*A_3\}_{i=1}^D, \{A_2A_i^* - A_i^*A_2\}_{i=1}^D,$
$\{AA_i^* - A_i^*A\}_{i=1}^D.$

The bilinear form $\langle\ ,\ \rangle$

We endow $\mathrm{Mat}_X(\mathbb{C})$ with the Hermitean inner product $\langle\ ,\ \rangle$ such that $\langle R, S\rangle = tr(R^t\overline{S})$ for all $R, S \in \mathrm{Mat}_X(\mathbb{C})$.
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Lemma 5.

(See [11, Lemma 3.2].) *For* $0 \le h, i, j, r, s, t \le D$,

(i) $\langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \rangle = \delta_{ir}\delta_{js}\delta_{ht}k_h p_{ij}^h$,

(ii) $\langle E_i A_j^* E_h, E_r A_s^* E_t \rangle = \delta_{ir}\delta_{js}\delta_{ht}m_h q_{ij}^h$.

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Lemma 6.

For $0 \leq h, i, j, r, s, t \leq D$ *we have*

$$\langle A_i E_j^* A_h, A_r E_s^* A_t\rangle = \sum_{\ell=0}^{D} k_\ell p_{ir}^\ell p_{js}^\ell p_{ht}^\ell .$$

Definition 7.

Let G denote the matrix of inner products for

$$A_2E_i^*A - AE_i^*A_2, A_3E_i^* - E_i^*A_3, A_2E_i^* - E_i^*A_2, AE_i^* - E_i^*A,$$

where $1 \leq i \leq D$. Thus the matrix G is $4D \times 4D$.

Theorem 8.

The entries of G are as follows: For $1 \leq i, j \leq D$,

$\langle \, , \, \rangle$	$A_2E_j^*A - AE_j^*A_2$	$A_3E_j^* - E_j^*A_3$	$A_2E_j^* - E_j^*A_2$	$AE_j^* - E_j^*A$
$A_2E_i^*A - AE_i^*A_2$	ϕ	$2k_2b_2(p_{ij}^1 - p_{ij}^2)$	$2k_2a_2(p_{ij}^1 - p_{ij}^2)$	$2k_2c_2(p_{ij}^1 - p_{ij}^2)$
$A_3E_i^* - E_i^*A_3$	$2k_2b_2(p_{ij}^1 - p_{ij}^2)$	$2k_3(\delta_{ij}k_i - p_{ij}^3)$	0	0
$A_2E_i^* - E_i^*A_2$	$2k_2a_2(p_{ij}^1 - p_{ij}^2)$	0	$2k_2(\delta_{ij}k_i - p_{ij}^2)$	0
$AE_i^* - E_i^*A$	$2k_2c_2(p_{ij}^1 - p_{ij}^2)$	0	0	$2k(\delta_{ij}k_i - p_{ij}^1)$

where $\phi/2$ is a weighted sum involving the following terms and coefficients:

term	*coefficient*
p_{ij}^0	kk_2
p_{ij}^1	$k_2a_1a_2 - kb_1^2$
p_{ij}^2	$k_2(c_2(b_1 - 1) - a_2(a_1 + 1) + b_2(c_3 - 1))$
p_{ij}^3	$-k_3c_3^2$

Definition 9.

For $1 \leq i \leq D$ let B_i denote the matrix of inner products for

$$A_2 A_i^* A - A A_i^* A_2,\ A_3 A_i^* - A_i^* A_3,\ A_2 A_i^* - A_i^* A_2,\ A A_i^* - A_i^* A.$$

So the matrix B_i is 4×4.

Definition 9.

For $1 \leq i \leq D$ let B_i denote the matrix of inner products for

$$A_2A_i^*A - AA_i^*A_2,\ A_3A_i^* - A_i^*A_3,\ A_2A_i^* - A_i^*A_2,\ AA_i^* - A_i^*A.$$

So the matrix B_i is 4×4.

Definition 10.

Let $\widetilde{G}$ denote the matrix of inner products for

$$A_2A_i^*A - AA_i^*A_2,\ A_3A_i^* - A_i^*A_3,\ A_2A_i^* - A_i^*A_2,\ AA_i^* - A_i^*A,$$

where $1 \leq i \leq D$. Thus the matrix $\widetilde{G}$ is $4D \times 4D$.

Lemma 11.

The matrix $\widetilde{G}$ *has the form*

$$\widetilde{G} = \begin{bmatrix} B_1 & & & \mathbf{0} \\ & B_2 & & \\ & & \ddots & \\ \mathbf{0} & & & B_D \end{bmatrix},$$

where $B_1, B_2, ..., B_D$ *are from Definition* 9.

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where $B_1, B_2, ..., B_D$ *are from Definition* 9.

Lemma 12.

(i) $det(\widetilde{G}) = \prod_{i=1}^{D} det(B_i)$.

(ii) $\widetilde{G} = (S')^t G S'$.

(iii) $det(G) = \big(det(S')\big)^{-2} det(\widetilde{G})$.

(iv) $det(G) = \big(det(S^{alt})\big)^{-8} \prod_{i=1}^{D} det(B_i)$.

The main result

For $1 \leq i \leq D$ let Γ_i denote the graph with vertex set X where vertices are adjacent in Γ_i whenever they are at distance i in Γ.

The graph Γ is said to be primitive whenever Γ_i is connected for $1 \leq i \leq D$.

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Theorem 13.

Let Γ denote a primitive distance-regular graph with diameter $D \geq 3$. Then Γ is Q-polynomial if and only if $det(G) = 0$.

In Theorem 13 we assume Γ is primitive in order to invoke Terwilliger's dual distance matrix characterization of the Q-polynomial property [10, Theorem 3.3].

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Thank you for your attention.

The main result

Lemma 14.

(See [BCN, Proposition 4.4.7].)
Assume Γ is primitive. Then $u_i(\theta_j) \neq 1$ for $1 \leq i, j \leq D$.

Theorem 15.

(See [Terwilliger 1995, Theorem 3.3].)
Let Γ denote a distance-regular graph with diameter $D \geq 3$, and let θ denote any eigenvalue of Γ. Let E denote the corresponding primitive idempotent, with dual eigenvalue sequence $\theta_0^*, \theta_1^*, ..., \theta_D^*$. Then $A^*A_3 - A_3A^* \in Span\{AA^*A_2 - A_2A^*A, A^*A_2 - A_2A^*, A^*A - AA^*\}$ if and only if Γ is Q-polynomial with respect to E.

The main result

Proof

($\Rightarrow$) Assume that Γ is Q-polynomial with respect to the ordering $E_0, E_1, ..., E_D$. Write $A^* = A^*_1$.

By Theorem 15 and Lemma 14, we obtain

$A^*A_3 - A_3A^* \in Span\{AA^*A_2 - A_2A^*A, A^*A_2 - A_2A^*, A^*A - AA^*\}$.

Thus $AA^*A_2 - A_2A^*A, A^*A_3 - A_3A^*, A^*A_2 - A_2A^*, A^*A - AA^*$ are linearly dependent.

Now the matrix B_1 from Definition 9 has determinant zero.

Now $det(G) = 0$ by Lemma 12(iii).

Proof

$(\Leftarrow)$ Assume $det(G) = 0$.
By Lemma 12(iii) and since S^{alt} is invertible, there exists an integer t $(1 \le t \le D)$ such that $det(B_t) = 0$.
Now $AA_t^*A_2 - A_2A_t^*A, A_t^*A_3 - A_3A_t^*, A_t^*A_2 - A_2A_t^*, A_t^*A - AA_t^*$ are linearly dependent.
We will show that
$A_t^*A_3 - A_3A_t^* \in Span\{AA_t^*A_2 - A_2A_t^*A, A_t^*A_2 - A_2A_t^*, A_t^*A - AA_t^*\}$.
To do this we show that $AA_t^*A_2 - A_2A_t^*A, A_t^*A_2 - A_2A_t^*, A_t^*A - AA_t^*$ are linearly independent.
Suppose not.
Then there exist scalars a, b, c, not all zero, such that

$$a(AA_t^*A_2 - A_2A_t^*A) + b(A_t^*A_2 - A_2A_t^*) + c(A_t^*A - AA_t^*) = 0. \qquad (9)$$

Abbreviate $\theta_i^* = m_t u_i(\theta_t)$ $(0 \le i \le D)$. So $A_t^* = \sum_{i=0}^{D} \theta_i^* E_i^*$.
By Lemma 14,

$$\theta_i^* \ne \theta_0^* \qquad (1 \le i \le D). \qquad (10)$$

Proof

For $1 \leq h \leq 3$ pick $z \in X$ such that $\partial(x,z) = h$.
Compute the (x,z)-entry in (9).
For $h = 3$ we get $ac_3(\theta_1^* - \theta_2^*) = 0$.
For $h = 2$ we get $aa_2(\theta_1^* - \theta_2^*) + b(\theta_0^* - \theta_2^*) = 0$.
For $h = 1$ we get $ab_1(\theta_1^* - \theta_2^*) + c(\theta_0^* - \theta_1^*) = 0$.
Solving these equations we obtain $a(\theta_1^* - \theta_2^*) = 0$ and $b = 0$, $c = 0$.
Recall that a, b, c are not all zero, so $a \neq 0$ and $\theta_1^* = \theta_2^*$.
Now (9) becomes $AA_t^*A_2 - A_2A_t^*A = 0$.
Recall $c_2A_2 = A^2 - a_1A - kI$.
We have $AA_t^*A^2 + kA_t^*A = A^2A_t^*A + kAA_t^*$.
Thus $[A, AA_t^*A + kA_t^*] = 0$.
For $0 \leq i,j \leq D$ such that $i \neq j$ we have $E_iA_t^*E_j(\theta_i\theta_j + k) = 0$.
Recall $E_iA_j^*E_h = 0$ if and only if $q_{ij}^h = 0$ for $0 \leq h,i,j \leq D$.
So $E_iA_t^*E_j \neq 0$ if and only if $q_{ij}^t \neq 0$, and in this case $\theta_i\theta_j + k = 0$.
Since $q_{0t}^t = 1$ and $\theta_0 = k$, we have $k\theta_t + k = 0$ and hence $\theta_t = -1$.

Proof.

Define a diagram with nodes $0, 1, ..., D$.
There exists an arc between nodes i, j if and only if $i \neq j$ and $q^t_{ij} \neq 0$.
Observe that node 0 is connected to node t and no other nodes.
By [BCN, Proposition 2.11.1] and Lemma 14, the diagram is connected.
Thus there exists a node s with $s \neq 0$ and $s \neq t$ that is connected to node t by an arc.
In other words $q^t_{st} \neq 0$.
So $\theta_s \theta_t + k = 0$ and hence $\theta_s = k$, a contradiction.
Therefore $AA^*_t A_2 - A_2 A^*_t A, A^*_t A_2 - A_2 A^*_t, A^*_t A - AA^*_t$ are linearly independent.
So
$A^*_t A_3 - A_3 A^*_t \in Span\{AA^*_t A_2 - A_2 A^*_t A, A^*_t A_2 - A_2 A^*_t, A^*_t A - AA^*_t\}$.
Now by Theorem 15 and (10), Γ is a Q-polynomial with respect to $E = E_t$. □

- E. Bannai, T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings, London, 1984.
- A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-regular graphs*, Springer-Verlag, Berlin, Heidelberg, 1989.
- E. Hanson, *A characterization of Leonard pairs using the parameters* $\{a_i\}_{i=0}^{d}$, Linear Algebra Appl. 438 (2013), 2289–2305.
- T. Ito, K. Tanabe, and P. Terwilliger, *Some algebra related to P- and Q-polynomial association schemes*, In Codes and Association Schemes (Piscataway NJ, 1999), Amer. Math. Soc., Providence RI, 2001, pp.167–192; arXiv:math. CO/0406556.
- A. Jurišić, P. Terwilliger, and A. Žitnik, *The Q-polynomial idempotents of a distance-regular graph*, J. Combin. Theory Ser. B 100 (2010), 683–690.
- H. Kurihara and H. Nozaki, *A characterization of Q-polynomial association schemes*, J. Combin. Theory Ser. A 119 (2012), 57–62.

K. Nomura and P. Terwilliger, *Tridiagonal matrices with nonnegative entries*, Linear Algebra Appl. 434 (2011), 2527–2538.

A.A. Pascasio, *A characterization of Q-polynomial distance-regular graphs*, Discrete Math. 308 (2008), 3090–3096.

P. Terwilliger, *A characterization of P- and Q-polynomial association schemes*, J. Combin. Theory Ser. A 45 (1987), 8–26.

P. Terwilliger, *A new inequality for distance-regular graphs*, Discrete Math. 137 (1995), 319–332.

P. Terwilliger, *The subconstituent algebra of an association scheme I*, J. Algebraic Combin. 1 (1992), 363–388.