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Eigenvalues and eigenvectors of the perfect matching association scheme

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\( \mathcal{M}_{2n} = \) set of all \textbf{perfect matchings} in the complete graph \( K_{2n} \).

\[ |\mathcal{M}_{2n}| = (2n - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n - 1). \]

\( S_{2n} \) has a natural substitution action on \( \mathcal{M}_{2n} \):

\[
\pi \cdot (\{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}) = \{\{\pi(i_1), \pi(j_1)\}, \ldots, \{\pi(i_n), \pi(j_n)\}\}
\]

\( \mathcal{B}_{2n} = \text{algebra} \) of all complex \( \mathcal{M}_{2n} \times \mathcal{M}_{2n} \) matrices commuting with the \( S_{2n} \) action on \( \mathcal{M}_{2n} \), i.e., for a \( \mathcal{M}_{2n} \times \mathcal{M}_{2n} \) matrix \( N \)

\( N \in \mathcal{B}_{2n} \) iff \( N \) is constant on the \( S_{2n} \)-orbits of \( \mathcal{M}_{2n} \times \mathcal{M}_{2n} \).
$S_{2n}$-orbits of $\mathcal{M}_{2n} \times \mathcal{M}_{2n}$

**Even partitions** of $2n$ are partitions of $2n$ with all parts even. They look like $2\lambda = (2\lambda_1, \ldots, 2\lambda_k)$ where $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

Let $A, B \in \mathcal{M}_{2n}$. Think of $A$ as the red edges and $B$ as the blue edges. Consider $A \cup B$. Each component is an even cycle and the number of vertices of the components gives an even partition $d(A, B) \vdash 2n$.

**Lemma** $(A, B), (C, D) \in \mathcal{M}_{2n} \times \mathcal{M}_{2n}$ are in the same $S_{2n}$-orbit if and only if $d(A, B) = d(C, D)$. Moreover, $(A, B), (B, A)$ are in the same $S_{2n}$-orbit, for all $(A, B) \in \mathcal{M}_{2n} \times \mathcal{M}_{2n}$. 

\{N(2\mu) : \mu \vdash n\} is the orbital basis of \(B_{2n}\), where \(N(2\mu) \in B_{2n}\) is the symmetric 0,1 matrix with entry in row \(A\), column \(B\) equal to 1 iff \(d(A, B) = 2\mu\). So \(\dim(B_{2n}) = p(n)\), the number of partitions of \(n\).

\(B_{2n}\) is an algebra of symmetric matrices, so is commutative and thus the \(S_{2n}\)-module \(\mathbb{C}[\mathcal{M}_{2n}]\) is multiplicity free. \(B_{2n}\) is called the Bose-Mesner algebra of the perfect matching association scheme.

What are the \(p(n)\) (common) eigenspaces of \(B_{2n}\)? For \(\lambda \vdash n\), let \(V^\lambda\) denote the \(S_n\)-irreducible parametrized by \(\lambda\).

**Fundamental Theorem** \(\mathbb{C}[\mathcal{M}_{2n}] \cong \bigoplus_{\lambda \vdash n} V^{2\lambda}\), as \(S_{2n}\)-modules.

Many proofs. Short proof in *James and Kerber* and *Saxl*.
So both the orbital basis and the eigenspaces of $B_{2n}$ are indexed by even partitions of $2n$. For $\lambda, \mu \vdash n$, define

$$\theta_{2\mu}^{2\lambda} = \text{eigenvalue of } N(2\mu) \text{ on } V^{2\lambda} \text{ (easily shown to be an integer)}.$$ 

We shall now describe our main results: one on eigenvalues and one on eigenvectors.

**Notational convention** Let $\mu \vdash m$ with all parts $\geq 2$. For $n \geq m$ we can consider $\mu$ as a partition of $n$ by adding 1 (the trivial part) $n - m$ times. We shall write $(\mu, 1^{n-m})$ for this partition but pronounce it as $\mu'$. 
Eigenvalues of $B_{2n}$

Motivation In studying a natural Markov chain on perfect matchings, Diaconis and Holmes considered the matrix

$$N((4, 2^{n-2})) = N(2(2, 1^{n-2}))$$

and gave a universal formula for $\theta_{2(2, 1^{n-2})}^{2\lambda}$, $\lambda \vdash n$.

We generalize this as follows:

Fix a partition $\mu \vdash m$ with all parts $\geq 2$. We give an algorithm that produces a universal formula for $\theta_{2(\mu, 1^{n-m})}^{2\lambda}$, $\lambda \vdash n \geq m$. 
Eigenvectors of $B_{2n}$

**Motivation** During the course of giving an algebraic proof of the Erdős-Ko-Rado theorem on intersecting families of perfect matchings, Godsil and Meagher write down an eigenvector (using a quotient argument) belonging to the eigenspace $V^{(2n-2,2)}$.

We generalize this by giving an inductive procedure to write down an eigenvector in each of the eigenspaces.

**Rest of the talk** Universal formula for $\theta_{2(\mu,1^n-m)}^{2\lambda}$. 
For \( \lambda, \mu \vdash n \),

\[ C_\mu = \text{conjugacy class in } S_n \text{ of cycle type } \mu \text{ and } c_\mu = \sum_{\pi \in C_\mu} \pi \in \mathbb{C}[S_n], \]

\[ \chi^\lambda = \text{character of } V^\lambda, \quad \chi^\lambda_\mu = \chi^\lambda(\pi), \quad \pi \in C_\mu, \]

\[ \phi^\lambda_\mu = \text{eigenvalue of } c_\mu \text{ on } V^\lambda = \frac{|C_\mu| \chi^\lambda_\mu}{\dim(V^\lambda)}, \]

\( \phi^\lambda_\mu \) is called a \textbf{central character} \textit{(can easily be shown to be an integer)}. 
$\Lambda =$ Symmetric functions in $\{x_1, x_2, x_3, \ldots \}$ with coefficients in $\mathbb{Q}[t]$.

Power sum symmetric function: $p_0 = 1$ and $p_n = \sum_i x_i^n$, $n \geq 1$.

$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ if $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$ (= all partitions).

The set $\{p_\lambda : \lambda \in \mathcal{P}\}$ is a $\mathbb{Q}[t]$-module basis of $\Lambda$.

The content $c(b)$ of a box $b$ of a Young diagram is

$$y$$-coordinate of box $b - x$$-coordinate of box $b$.

Contents in the first row are $0, 1, 2, \ldots$, second row are $-1, 0, 1, 2, \ldots$ and so on.
Given \( f \in \Lambda \) and \( \lambda \in \mathcal{Y}_n \) (= Young diagrams with \( n \) boxes) we define the **content evaluation** \( f(c(\lambda)) \) to be the rational number obtained from \( f \) by setting \( t = n \), \( x_i = 0 \) for \( i > n \), and \( \{x_1, \ldots, x_n\} \) to the multiset of the contents of the \( n \) boxes of \( \lambda \).

**Examples of universal formula**

**Frobenius** \( \phi^\lambda_{(2,1^{n-2})} = p_1(c(\lambda)) = \text{Sum of all contents} \), \( \lambda \in \mathcal{Y}_n \).

**Ingram** \( \phi^\lambda_{(3,1^{n-3})} = (p_2 - \frac{t(t-1)}{2})(c(\lambda)) \)

\[= \text{Sum of squares of all contents} - \frac{n(n-1)}{2}, \lambda \in \mathcal{Y}_n.\]

There is a vast generalization. Let \( \mathcal{P}(2) \) denote the set of partitions with all parts \( \geq 2 \).
Thm. 1 (Corteel, Goupil, and Schaeffer (2004) and Garsia (2003))

For each $\mu \in \mathcal{P}(2)$ there is an algorithm to compute a symmetric

function $W_\mu \in \Lambda$ such that

(i) $\{W_\mu : \mu \in \mathcal{P}(2)\}$ is a $\mathbb{Q}[t]$-module basis of $\Lambda$.

(ii) Let $\mu \in \mathcal{P}(2)$ with $|\mu| = m$. Let $\lambda \in \mathcal{P}$ with $|\lambda| = n \geq m$. We have

$$W_\mu(c(\lambda)) = \phi^\lambda_{(\mu,1^{n-m})}.$$ 

Example ($|\mu| \leq 4$). This gives the character tables of $S_1, \ldots, S_4$ and the

first five characters of $S_n, \ n \geq 5$.

$W_\emptyset = 1$ \quad $W_2 = p_1$ \quad $W_3 = p_2 - \frac{t(t-1)}{2}$

$W_{2,2} = \frac{p_1^2}{2} - \frac{3p_2}{2} + \frac{t(t-1)}{2}$ \quad $W_4 = p_3 - (2t - 3)p_1$
Thm. 2 For each \( \mu \in \mathcal{P}(2) \) there is an algorithm to compute a symmetric function \( E_\mu \in \Lambda \) such that

(i) \( \{E_\mu : \mu \in \mathcal{P}(2)\} \) is a \( \mathbb{Q}[t] \)-module basis of \( \Lambda \).

(ii) Let \( \mu \in \mathcal{P}(2) \) with \( |\mu| = m \). Let \( \lambda \in \mathcal{P} \) with \( |\lambda| = n \geq m \). We have

\[
E_\mu(c(2\lambda)) = \theta^2_{2(\mu,1^m)}.
\]

Example (\( |\mu| \leq 4 \)) This gives the eigenvalues of \( B_2, \ldots, B_8 \) and the first five eigenvalues of \( B_{2n}, n \geq 5 \). \( E_\emptyset = 1 \) and

\[
\begin{align*}
E_2 &= \frac{p_1}{2} - \frac{t}{4} \\
E_2,2 &= \frac{p_1^2}{8} - \frac{3p_2}{4} + \frac{(10-t)p_1}{8} + \frac{9t^2-24t}{32} \\
E_3 &= \frac{p_2}{2} - p_1 + \frac{3t-t^2}{4} \\
E_4 &= \frac{p_3}{2} - \frac{9p_2}{4} + \frac{(11-2t)p_1}{2} + \frac{8t^2-23t}{8}
\end{align*}
\]
Two ingredients in the **Proof of Theorem 2**: Theorem 1 plus a **combinatorial algorithm** that, starting with the central characters of \(S_2, S_4, \ldots, S_{2n}\) produces the eigenvalues of \(\mathcal{B}_2, \mathcal{B}_4, \ldots, \mathcal{B}_{2n}\) by solving linear equations of size at most \(p(n - 1) \times p(n - 1)\).

Moreover, given \(\{W_\mu \mid \mu \leq m\}\) this algorithm produces \(\{E_\mu \mid \mu \leq m\}\) by solving linear equations of size \(p(m - 1) \times p(m - 1)\).

**Ongoing project Garsia’s** paper contains a list of the symmetric functions \(W_\mu\) for \(|\mu| \leq 8\). Since \(p(7) = 15\) the algorithm above needs to solve equations of size at most \(15 \times 15\) to produce \(E_\mu\) for \(|\mu| \leq 8\). This seems **feasible**, yielding eigenvalues of \(\mathcal{B}_2, \ldots, \mathcal{B}_{16}\).
THANK YOU