All About the CFSG

Ron Solomon

The Ohio State University

Delaware Conference, August 2017

A friend of mine recently received a referee's report containing the following statement:

"The classification of finite simple groups will soon become (if it is not already the case) a mathematical realm that no one understands the proof of ... "

So, my task in this lecture is to unlock this "Book with Seven Seals".

I interpret "understands the proof" as a weaker statement than "has checked every line of the proof." This is indeed a work-in-progress, and some small gaps continue to be found by Lyons and myself.

My goal is that of Rabbi Hillel: to say enough to enable and inspire some of you to study the proof more carefully yourselves, starting with the more detailed Guide to the Perplexed provided by ALSS.

As I tell my students: In trying to do a proof, first write down the hypotheses as line 1, Then skip down a ways and write the conclusion. Then fill in the gap.

Line 1: Let G be a nonabelian finite simple group.

....

Line 1,000,000: Hence G is isomorphic to an alternating group, a group of Lie type, or one of 26 sporadic simple groups.

Okay. now we can add Line 2: Assume G is a minimal counterexample to Line 1,000,000.

Now, by induction we know all possible composition factors for all proper subgroups of G.

Getting Line 999,999 is a bit trickier. We need a strategy for identifying G as one of the groups listed on Line 1,000,000. For simplicity (in the English, not the French sense), let's assume that the only conclusion is $G = PSL(n, q)$ for some $n \geq 2$ and q a prime power.

There are two principal identification methods:

Method A: The unipotent (or parabolic) method (Tits): The cosets of the parabolic subgroups of $PSL(n, q)$ define a projective geometry of dimension $n - 1$ on which G acts, and indeed G is identified as "most" of the automorphism group of this geometry (when $n \geq 3$).

More generally, the geometry is a Tits building.

Method B: The semisimple method (Steinberg-Curtis-Tits-Phan): A family $(L_1, \dots, L_{n-1})$ of Levi complements of minimal parabolics, together with their canonical embeddings in the rank 2 groups $\langle L_i, L_j \rangle \cong SL_3(q)$ defines a unique amalgam of groups whose completion is $SL(n, q)$ and G is identified as a central quotient of this amalgam, for $n \geq 4$.

The uniqueness of the amalgam is a recent theorem of Blok, Hoffman and Shpectorov. The identification of the completion of the amalgam goes back to work of Curtis and Tits. Later, Phan, and more recently Bennett, Shpectorov, Gramlich and others, proved variant recognition theorems in which, for example, $\langle L_i, L_j \rangle \cong SU_3(q)$ instead of $SL_3(q)$.

If $G = PSL(n, p^m)$, then a parabolic subgroup of G is the normalizer of a non-trivial normal subgroup of a Sylow p-subgroup U of G. The elements of U are of the form $I + N$, where N is a nilpotent matrix; hence $I + N$ is a unipotent matrix. Anyway, to implement Method A, we must identify the normalizers of certain unipotent subgroups of G, i.e., p-subgroups where p is the characteristic prime. These normalizers are called the p-local subgroups of G.

One problem: An unknown simple group G doesn't come with a "characteristic prime" identified. I'll address that problem shortly.

If $G = PSL(n, p^m)$, then a Levi complement L can usually be observed in the centralizer of a semisimple (diagonalizable) element of G, e.g. if p and n are odd, and $t = diag(-1, \ldots, -1, 1)$, then $C_{SL(n,p^m)}(t) \cong GL(n-1, p^m)$ contains a family $(L_1, \ldots, L_{n-2})$ of minimal Levi complements, while $s = diag(1, -1, \ldots, -1)$ contains $(L_2, \ldots, L_{n-1})$, and all Curtis-Tits relations except for $[L_1, L_{n-1}] = 1$ visibly hold.

Companion problem: "Diagonalizable" makes no sense when G is an unknown simple group.

Key observation: Unipotent and semisimple elements of $G = PSL(n, p^m)$ (or any group of Lie type) can be distinguished by their centralizers, namely:

If x is unipotent (e.g., of order p), then every minimal normal subgroup of $C_G(x)$ is a p-group.

If r is a semisimple prime, then for most elements x of order r, $C_G(x)$ has a subnormal (often, normal) subgroup L which is quasisimple, i.e., $L = [L, L]$ and $L/Z(L)$ is a non-abelian simple group. Such an L is called a **component** of $C_G(x)$.

Now, most primes are odd. So, looking in the back of the book, we see that 2 is a semisimple prime for most finite simple groups — almost all groups of Lie type defined over fields of odd order and also all alternating groups A_n for $n \geq 9$. Hence, to pursue Method B, it is natural to study centralizers of elements of order 2 (involutions). This is the method that was proposed by Richard Brauer in his ICM address in 1954. There was one problem: G might have odd order. This problem was solved in 1963:

Theorem

(Feit-Thompson) There is no non-abelian simple group of odd order.

Now that we are guaranteed elements of order 2 in G, we may subdivide the problem into two cases as follows:

A) G is of characteristic 2-type: For every involution t of G, every minimal normal subgroup of $C_G(t)$ is a 2-group, OR
B) G is of odd type: G is not of characteristic 2-type.

First we would like to prove:

If G is of odd type, then G is of component type for 2, i.e., for at least one involution t of G, $C_G(t)$ has a component, i.e. a quasisimple subnormal subgroup L.

Problem: That statement is false.

Example: $G = PSL(2, q)$, q odd, $q \geq 11$.

For, if $q \equiv \epsilon \pmod 4$, then $C_G(t) \cong D_{q-\epsilon}$ has no component (it is solvable), but it has a cyclic normal subgroup of odd order $(q - \epsilon)/4$.

Good news: This problem only happens when G has 2-rank ≤ 2, i.e., when G contains no subgroup isomorphic to $C_2 \times C_2 \times C_2$.

These “small”simple groups present special difficulties. For one things, distinct subgroups tend to have very small (often trivial) intersections. This undermines one of the key techniques of the field: Studying the intersection of distinct subgroups, using refinements of the following naive observation:

Let M and N be distinct maximal subgroups of the simple group G. If $1 \neq X \leq M \cap N$, then X cannot be normal in both M and N, since if it were, then X would be normal in $G = \langle M, N \rangle$.

The good news is that when distinct subgroups have very small intersections, it is realistic to try to count the elements of G. Somewhat more subtly, you can try to do character theory. This method was pioneered by Frobenius who solved an important case of this problem in 1900:

Theorem

(Frobenius) Let G be a finite group with a proper subgroup H such that $H \cap H^g = 1$ for all $g \in G - H$. Then G is not a simple group.

The methods of Frobenius were exploited by Brauer, Suzuki and others in the 1950s. One outcome was:

Theorem

(Strongly Embedded Theorem of Suzuki-Bender (1968)) Let G be a finite simple group with a proper subgroup H of even order such that $H \cap H^g$ has odd order for all $g \in G - H$. Then $G \cong PSL(2, 2^n)$ or $PSU(3, 2^n)$ or G is a Suzuki group, $Sz(2^{2n+1})$.

AN ASIDE: Character theory has played an essential but very limited role in the CFSG, mostly in the Feit-Thompson Theorem and the Suzuki-Bender Theorem. Possibly there is a larger role. Here is a problem where character theory might play a role.

PROBLEM: Let G be a finite group with an elementary abelian Sylow p-subgroup A. Suppose that B is a subgroup of A such that $b^G \cap A \leq B$ for all $b \in B$. Prove that G has a normal subgroup N with $B \in Syl_p(N)$.

Note: This statement is a corollary of the CFSG, and, for $p = 2$, it is a theorem of Goldschmidt. The challenge is to prove it for odd primes p without assuming the CFSG.

Also, we have

Theorem

(Gorenstein-Walter, Alperin-Brauer-Gorenstein (1970)) Suppose that G *is a finite simple group of odd type, but not of component type. If* G *has 2-rank* ≤ 2*, then either* $G \cong PSL(2,q)$ *for some odd* $q \geq 11$*, or* $G \cong A_7$*.*

Note: The final identification of $PSL(2,q)$ is a problem, since the geometry of the projective line is trivial. The group must be identified as a permutation group. This was done by Zassenhaus in the 1930s.

When G has 2-rank ≥ 3, the 2-Signalizer Functor Theorem of Gorenstein and Goldschmidt may be used to prove:

Theorem

(The B-Theorem) Let G be a finite simple group of odd type and of 2-rank ≥ 3. Then G is of component type; i.e.:
For some involution t of G, $C_G(t)$ has a component, L.

The proof of this theorem is rather intricate, but it is really fairly easy, except in troublesome "small" cases. This is a continual refrain throughout the proof of the CFSG.

Continuing our study of the simple groups of odd type, we reach a crucial theorem of Michael Aschbacher:

Theorem

(Standard Component Theorem) Let G be a finite simple group of component type. Then (with one small exception) G contains a standard subgroup L, i.e., L is a normal quasisimple subgroup of some involution centralizer, and $Q := C_G(L)$ is very small.

We look at the set $\mathcal{L}$ of all components of all involution centralizers and choose $L \in \mathcal{L}$ with $|L|$ as large as possible. Aschbacher calls Q a tightly embedded subgroup of G. It can be shown that Q has 2-rank 1 or $|Q| = 4$. The "one small exception" arises when $L \cong SL_2(q)$.

At this point we are almost done with Case B. If the target group $G = PSL(n, q)$, we typically find $L \cong SL(n-1, q)$ and a G-conjugate L^g with $L \cap L^g \cong SL(n-2, q)$. Then we read off a set of Curtis-Tits generators and relations which identify G. Truthfully, we read off all but one relation, and for small n, this is a problem. For example, if $L \cong SL(6, q)$, we could have $G \cong E_6(q)$.

Now, assume that we are in Case A: G is of characteristic 2-type (the Even Case, for short). In the existing Classification Proof, as well as the GLS Revision, the strategy for the Even Case is to look for a semisimple prime p and repeat the Method B approach, replacing 2 with p. This works well except in two "extreme" cases:

Case Q: The Quasithin Case

In the Odd Case, the subcase when G had 2-rank ≤ 2 required special treatment. In this Even Case, the subcase when G has p-rank ≤ 2 for all odd primes p is a special problem. This requires the Unipotent Method A, and was handled by Aschbacher and Smith.

Case GF2: The Bicharacteristic Case

This is the case when G is of characteristic p-type for ALL primes p for which G has p-rank ≥ 3. In fact, this can never happen by a theorem of Klinger and G. Mason. However, a slightly more general case leads to several of the large sporadic simple groups, including the Monster, the Baby Monster, the Fischer groups and the two largest Conway groups. In all cases, except for Fi_{23}, we find an involution z in G such that $C_G(z)$ contains a large normal extraspecial 2-subgroup Q. (Large means $C_G(Q) = Z(Q)$.) A large body of literature by Janko, Fischer, Aschbacher, Timmesfeld, Smith and others characterizes these groups.

There is an alternative strategy for the Even Case, whose implementation is part of an ongoing project led by Meierfrankenfeld, Stellmacher and Stroth. The idea is to stick to the prime 2 and apply the Unipotent Method A in the Even Case. More specifically, one fixes a Sylow 2-subgroup T of G and studies the 2-local subgroups of G containing S. These should be the parabolic subgroups of G, if G is a simple group of Lie type in characteristic 2, as expected.

If there is a unique maximal 2-local subgroup of G containing T, then G should be a simple group of Lie type in characteristic 2 of BN-rank 1, i.e., one of the conclusions of the Suzuki-Bender Strongly Embedded Theorem. This is the content of the $C(G, T)$ Theorem of Aschbacher, which is also used in the proof of the Quasithin Theorem.

If there is a set $\{P_1, \ldots, P_r\}$ of maximal 2-local subgroups of G containing T with $r > 1$, then the cosets G/P_i define a Tits geometry Γ for G, which can be proven to satisfy the axioms for a Tits building, except in a few exceptional cases (such as one leading to the Thompson sporadic simple group, Th). G is then identified as a large normal subgroup of $Aut(\Gamma)$.

The MSS Project would replace the entire Even Case, including the Quasithin Case.

To each finite group, G, and Sylow p-subgroup P of G, there is associated a category, $\mathcal{F}_p(G)$, whose objects are the subgroups of P and whose morphisms are the functions $c_g : Q \to R$ induced by conjugations by elements g of G.

There is a well-defined notion of an (abstract) p-fusion system $\mathcal{F}_P$, based on a finite p-group P, and even a notion of a simple p-fusion system, $\mathcal{F}_P$. When $p = 2$, there is a conjecture:

CONJECTURE: Every simple 2-fusion system, $\mathcal{F}_P$, is the fusion system, $\mathcal{F}_P(G)$, for some finite simple group G, with one infinite family of exceptions, $\mathcal{F}_{Sol}(3^n)$, $n = 1, 2, \ldots$.

This conjecture is being actively pursued by Michael Aschbacher and a team of collaborators (Ellen Henke, Justin Lynd, etc.). If true, this would provide further evidence of the validity of the CFSG, as well as emphasize the importance of the prime 2 in the theory of finite simple groups.

THANK YOU ! AND ... CONGRATULATIONS ON A GREAT CAREER, ANDY !